Lepton number conservation and neutrino mixing in Quantum Field Theory

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Amplitudes of weak interaction processes $W^+ \to e^+ + \nu_e$ and $W^+ \to e^+ + \nu_\mu$, computed by using the QFT mixing formalism in the short time limit, are consistent with the lepton charge conservation expected in the tree level approximation of the Standard Model. In the same limit, use of the QM (Pontecorvo) formalism produces the violation of the lepton charge in the vertex.

In the long time limit, inconsistency with the Standard Model arises in both formalisms.

Neutrino mixing in QFT

Dirac fields with definite masses:

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i}\, \alpha^r_{\mathbf{k},i}(t) + v^r_{-\mathbf{k},i}\, \beta^{r\dagger}_{-\mathbf{k},i}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad i = 1,2.$$

The mixing relations:

$$\begin{aligned} \nu_e(x) &= \nu_1(x)\cos\theta + \nu_2(x)\sin\theta \\ \nu_\mu(x) &= -\nu_1(x)\sin\theta + \nu_2(x)\cos\theta \end{aligned}$$

can be written as*

$$\begin{aligned} \nu_e^\alpha(x) &= G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t) \\ \nu_\mu^\alpha(x) &= G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t) \end{aligned}$$

with unitary generator given by:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

- $G_\theta^{-1}(t)$ **maps** $\mathcal{H}_{1,2}$ **to** $\mathcal{H}_{e,\mu}$**:** $\qquad G_\theta(t) : \mathcal{H}_{1,2} \to \mathcal{H}_{e,\mu}.$

*M.Blasone and G.Vitiello, Annals Phys. (1995)

The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$; at finite volume:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\ |0\rangle_{1,2}$$

- **Orthogonality between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$ for $V \to \infty$**

- **Mass and flavor representations are unitary inequivalent for $V \to \infty$**

- **Time dependence:**

$$|0\rangle_{e,\mu} \equiv |0(0)\rangle_{e,\mu} = e^{-iHt}|0(t)\rangle_{e,\mu}$$

Condensate structure of $|0\rangle_{e,\mu}$ (use $\epsilon^r = (-1)^r$)

$$
\begin{aligned}
|0\rangle_{e,\mu} &= \prod_{\mathbf{k},r} [(1 - \sin^2\theta\, |V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}) \\
&+ \epsilon^r \sin^2\theta\, |V_{\mathbf{k}}||U_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) \\
&+ \sin^2\theta\, |V_{\mathbf{k}}|^2\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}]|0\rangle_{1,2}
\end{aligned}
$$

- 4 kinds of particle-antiparticle pairs with zero momentum and spin.

- **Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:**

$$\alpha^r_{\mathbf{k},\nu_e}(t) \;=\; \cos\theta\,\alpha^r_{\mathbf{k},1}(t) + \sin\theta\left(|U_{\mathbf{k}}|\,\alpha^r_{\mathbf{k},2}(t) + \epsilon^r\,|V_{\mathbf{k}}|\,\beta^{r\dagger}_{-\mathbf{k},2}(t)\right)$$

etc.. with $U_{\mathbf{k}}$, $V_{\mathbf{k}}$ Bogoliubov coefficients:

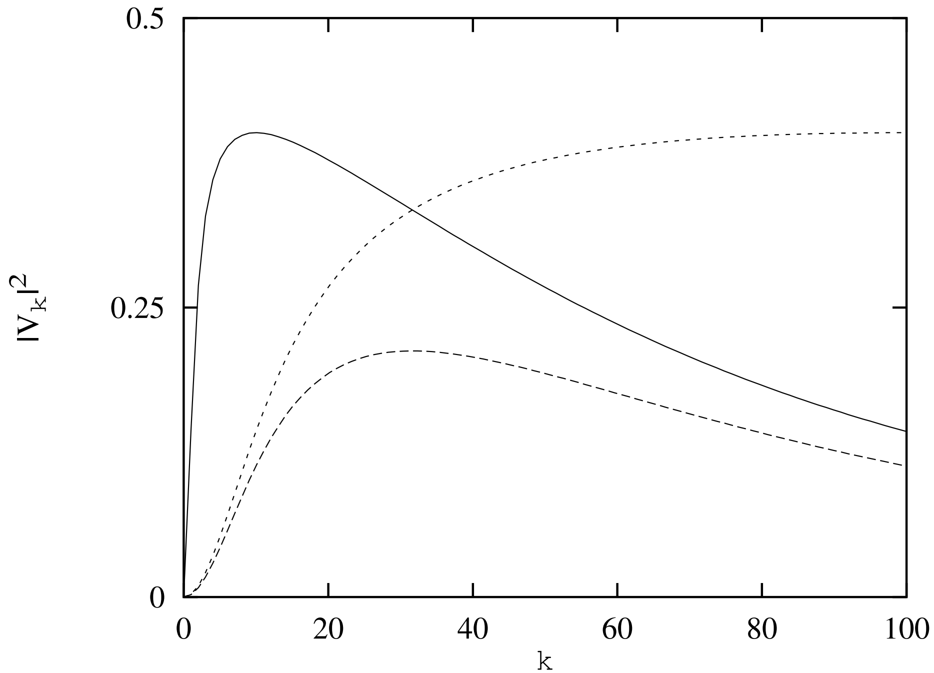
$$|U_{\mathbf{k}}| = u^{r\dagger}_{\mathbf{k},2}u^r_{\mathbf{k},1}\,, \qquad |V_{\mathbf{k}}| = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1}v^r_{-\mathbf{k},2}\,, \qquad |U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

$$\alpha_{\nu_e}(t)|0(t)\rangle_{e,\mu} = G^{-1}_\theta(t)\alpha_1(t)G_\theta(t)\; G^{-1}_\theta(t)|0\rangle_{1,2} = 0$$

- **The eigenstates of flavor charges:**

$$|\nu^r_{\mathbf{k},\sigma}(t)\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(t)|0(t)\rangle_{e,\mu}, \qquad \sigma = e, \mu.$$

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; **Long dashed line:** $m_1 = 10$, $m_2 = 100$; **Short dashed line:** $m_1 = 10$, $m_2 = 1000$.

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} =_{e,\mu} \langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} = \sin^2\theta\;|V_{\mathbf{k}}|^2, \qquad i = 1,2.$$

- $V_{\mathbf{k}} = 0$ **when** $m_1 = m_2$ **and/or** $\theta = 0$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2-m_1)^2}{4k^2}$ **for** $k \gg \sqrt{m_1 m_2}$.
- **Max. at** $k = \sqrt{m_1 m_2}$ **with** $V_{max} \to \frac{1}{2}$ **for** $\frac{(m_2-m_1)^2}{m_1 m_2} \to \infty$.

Neutrino oscillation formulae in QFT*:

$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t) \;=\; 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta)\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\,t\right) - |V_{\mathbf{k}}|^2 \sin^2(2\theta)\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\,t\right)$$

$$\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t) \;=\; |U_{\mathbf{k}}|^2 \sin^2(2\theta)\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\,t\right) + |V_{\mathbf{k}}|^2 \sin^2(2\theta)\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\,t\right)$$

- **Correction to amplitudes + new oscillating term**

- **For** $k \gg \sqrt{m_1 m_2}$ **we have:** $|V_{\mathbf{k}}|^2 \to 0$ **and** $|U_{\mathbf{k}}|^2 \to 1$ **the Pontecorvo formulae are reobtained in the relativistic limit.**

- **Similar results for three flavor neutrino fields and for boson fields**†

*M.Blasone, P.Henning and G.Vitiello, Phys. Lett. B (1999)

†M.Blasone, A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. D (2001)
M.Blasone, A.Capolupo and G.Vitiello, Phys. Rev. D (2002)
A.Capolupo, C.R.Ji, Y.Mischenko and G.Vitiello, Phys. Lett.B (2004)
M.Blasone, A.Capolupo, F.Terranova and G.Vitiello, Phys. Rev. D (2005)

Tree level approximation for the decays

$$W^+ \rightarrow e^+ + \nu_e \ ,$$
$$W^+ \rightarrow e^+ + \nu_\mu \ ,$$

where neutrinos are produced through charged current processes.

The Hamiltonian is

$$H_{int}(x) = -\frac{g}{\sqrt{2}} W_\mu^+(x) J_W^{\mu+}(x) + h.c. = -\frac{g}{2\sqrt{2}} W_\mu^+(x) \overline{\nu}_e(x) \gamma^\mu (1-\gamma^5) e(x) + h.c.$$

$W^+(x)$, $e(x)$, $\nu_e(x)$ **are the boson** W^+, **the electron and the flavor (electron) neutrino field, respectively.**

Assume the decays take place at time $t = x_I^0$, the amplitudes in the first order of perturbation theory are given by

$$A_{W^+ \to e^+ + \nu_e} = \langle \nu^r_{\mathbf{k},e}(x_I^0), e^s_{\mathbf{q}} | \left[-i \int_{x^0_{in}}^{x^0_{out}} d^4x \, H_{int}(x) \right] | W^+_{\mathbf{p},\lambda} \rangle$$

$$A_{W^+ \to e^+ + \nu_\mu} = \langle \nu^r_{\mathbf{k},\mu}(x_I^0), e^s_{\mathbf{q}} | \left[-i \int_{x^0_{in}}^{x^0_{out}} d^4x \, H_{int}(x) \right] | W^+_{\mathbf{p},\lambda} \rangle$$

QFT mixing formalism: Short time limit

Assuming $x^0_{in} = -\Delta t/2$ and $x^0_{out} = \Delta t/2$, for small Δt, i.e. $\Delta t = x^0_{out} - x^0_{in}$ much shorter than the characteristic neutrino oscillation time, we obtain in the first order of Δt,

$$A_{W^+ \to e^+ + \nu_e} \simeq \frac{i\,g}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k})\, \Delta t\, \overline{u}^r_{\mathbf{k},1}\, \gamma^\mu (1 - \gamma^5)\, v^s_{\mathbf{q},e}\,.$$

$$A_{W^+ \to e^+ + \nu_\mu} \simeq \frac{i\,g}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k})\, \sin\theta \cos\theta \times$$

$$\times \;\; \Delta t [\overline{u}^r_{\mathbf{k},2} - |U_{\mathbf{k}}|\, \overline{u}^r_{\mathbf{k},1} + \varepsilon^r\, |V_{\mathbf{k}}|\, \overline{v}^r_{-\mathbf{k},1}]\, \gamma^\mu (1 - \gamma^5)\, v^s_{\mathbf{q},e}\,.$$

Since

$$\overline{u}^r_{\mathbf{k},1}\, |U_{\mathbf{k}}| - \varepsilon^r\, \overline{v}^r_{-\mathbf{k},1} |V_{\mathbf{k}}| \;=\; \overline{u}^r_{\mathbf{k},2},$$

in the reference frame where $\mathbf{k} = (0, 0, |\mathbf{k}|)$, we get

$$A_{W^+ \to e^+ + \nu_\mu} \simeq 0\,.$$

as it should be according the SM phenomenology of the lepton charge conservation at the vertex.

QM mixing formalism: Short time limit

In the relativistic limit,for small Δt:

$$A^P_{W^+\to e^+ + \nu_e} \;\simeq\; \frac{i\,g}{2\sqrt{2}(2\pi)^{3/2}}\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\,\Delta t\,\overline{u}^r_{\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}$$

$$A^P_{W^+\to e^+ + \nu_\mu} \;\simeq\; -\frac{i\,g}{2\sqrt{2}(2\pi)^{3/2}}\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\;\sin\theta\cos\theta$$
$$\times\;\; \Delta t\,\frac{\Delta m}{2k}\,\overline{v}^r_{-\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,.$$

Since $A^P_{W^+\to e^+ + \nu_\mu}$ is not zero in the first order of Δt, this result shows that the use of QM formalism leads to a violation of the lepton charge in the production vertex. Same result is obtained for a representation introduced by Carlo Giunti (hep-ph/0402217).

QFT mixing formalism: Long time limit

In the scattering theory for finite range potentials, the interaction Hamiltonian $H_{int}(x)$ can be switched off adiabatically ($x^0_{in} \to -\infty$, $x^0_{out} \to +\infty$) so that initial and final states can be represented by eigenstates of the free Hamiltonian. We have

$$
\begin{aligned}
A_{W^+\to e^+ + \nu_e} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}}\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\,\{\overline{u}^r_{\mathbf{k},1}\,\gamma^\mu\,(1-\gamma^5)\,v^s_{\mathbf{q},e} \\
&\times\;[\cos^2\theta\,\delta(\omega_p-\omega_q-\omega_{k,1}) \\
&+\;\sin^2\theta\,\Big(|U_{\mathbf{k}}|^2\,\delta(\omega_p-\omega_q-\omega_{k,2})+|V_{\mathbf{k}}|^2\,\delta(\omega_p-\omega_q+\omega_{k,2})\Big)\,] \\
&+\;\varepsilon^r\,|U_{\mathbf{k}}||V_{\mathbf{k}}|\,\overline{v}^r_{-\mathbf{k},1}\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e}\sin^2\theta \\
&\times\;\Big[\delta(\omega_p-\omega_q+\omega_{k,2})-\delta(\omega_p-\omega_q-\omega_{k,2})\Big]\,\}
\end{aligned}
$$

and

$$
\begin{aligned}
A_{W^+ \to e^+ + \nu_\mu} &= \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}} \frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}} \, \delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}) \;\sin\theta\cos\theta \\
&\times \;\; [\overline{u}^r_{\mathbf{k},2}\, \gamma^\mu(1-\gamma^5)\, v^s_{\mathbf{q},e}\, \delta(\omega_p - \omega_q - \omega_{k,2}) \\
&- \;\; |U_\mathbf{k}|\, \overline{u}^r_{\mathbf{k},1}\, \gamma^\mu(1-\gamma^5)\, v^s_{\mathbf{q},e}\, \delta(\omega_p - \omega_q - \omega_{k,1}) \\
&+ \;\; \varepsilon^r\, |V_\mathbf{k}|\, \overline{v}^r_{-\mathbf{k},1} \gamma^\mu(1-\gamma^5) v^s_{\mathbf{q},e}\, \delta(\omega_p - \omega_q + \omega_{k,1})].
\end{aligned}
$$

which is not zero, in contrast with the conservation of lepton charge at the vertex predicted by the weak interaction Hamiltonian (which is by construction lepton charge conserving at tree level).

QM mixing formalism: Long time limit

The vacuum condensation effects are now excluded. We have:

$$A^P_{W^+\to e^+ + \nu_e} = \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}}\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}[\cos^2\theta\,\overline{u}^r_{\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,1})$$
$$+\ \sin^2\theta\,\overline{u}^r_{\mathbf{k},2}\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,2})]\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})$$

$$A^P_{W^+\to e^+ + \nu_\mu} = \frac{i\,g}{2\sqrt{2}(2\pi)^{1/2}}\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2\omega_p}}\sin\theta\,\cos\theta$$
$$\times\ [\overline{u}^r_{\mathbf{k},2}\,\gamma^\mu\,(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,2})$$
$$-\ \overline{u}^r_{\mathbf{k},1}\,\gamma^\mu\,(1-\gamma^5)\,v^s_{\mathbf{q},e}\,\delta(\omega_p-\omega_q-\omega_{k,1})]\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})$$

Similar result is obtained for the Giunti representation.

In the relativistic limit $|V_\mathbf{k}|\to 0$, $|U_\mathbf{k}|\to 1$, the QFT result coincides with the QM result.

QFT and QM amplitudes $A_{W^+\to e^+ + \nu_\mu}$ and $A^P_{W^+\to e^+ + \nu_\mu}$ are not zero.

In the decay processes where the mixed neutrinos are produced, the adiabatic hypothesis cannot be applied. The integration limits must be chosen so that $\Delta t = x^0_{out} - x^0_{in}$ is much shorter than the characteristic neutrino oscillation time.