Neutrino Mass and New Physics*

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Abstract

We review the present state of and future outlook for our understanding of neutrino masses and mixings. We discuss what we think are the most important perspectives on the plausible and natural scenarios for neutrinos and what may have the most promise to throw light on the flavor problem of quarks and leptons. We focus on the seesaw mechanism which fits into the big picture of particle physics such as supersymmetry and grand unification providing a unified approach to flavor problem of quarks and leptons. We argue that in combination with family symmetries, this may be at the heart of a unified understanding of flavor puzzle. We also discuss other new physics ideas such as neutrinos in models with extra dimensions and possible theoretical implications of sterile neutrinos. We outline some tests for the various schemes.

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Contents

I. INTRODUCTION

The standard model of electro-weak and strong interactions is an overwhelmingly successful theory for particles and forces. It probes physics below a hundred GeV's (in some cases up to several TeV's) and has met the challenge of many high precision experiments. There are however strong reasons from considerations of both particle physics as well as cosmology to suspect that there is a good deal of new physics beyond the standard model. Examples from cosmology are the need for dark matter, inflation as well as dark energy for a complete understanding of Big Bang Cosmology. On the side of particle physics, recent discovery of flavor conversion of solar, atmospheric, reactor and accelerator neutrinos have conclusively established that neutrinos have nonzero mass and they mix among themselves much like the quarks, thereby providing the first evidence of new physics beyond the standard model. They in turn have triggered enormous theoretical activity attempting to uncover the nature of this new physics. This includes further developments of the already existing mechanisms and theories such as GUT's and appearance of new ideas and approaches. Various aspects of these developments have to some extent been covered in several recent reviews [1]. The present review is an update which focuses on what we think are the most important perspectives on the most plausible and natural scenarios of physics beyond the standard model.

The main points stressed in this review are:

1). After all recent developments, the seesaw mechanism with large scale of the $B-L$ violations still looks as the most appealing and natural mechanism of neutrino mass generation. At the same time it is not excluded that some more complicated version of this mechanism is realized.

2). Grand Unification (plus supersymmetry in some form) still looks like the most preferable (plausible) scenario of physics which naturally embeds the seesaw.

3). At the same time the "seesaw - GUT" scenario does not provide complete understanding masses and mixing of neutrinos as well as masses and mixing of other fermions (or in other words, the flavor structure of the mass matrices). Some new physics on the top of this scenario seems essential.

In this connection two issues are of great importance:

- possible existence of new symmetries which show up mainly (only?) in the lepton sector;
- understanding relation between quarks and leptons.

To uncover this new additional element(s) of theory, the bottom-up approach is extremely important.

4). Alternative mechanisms and alternative pictures are not excluded, though they have not reached the same level of sophistication as the GUT approach and appear less plausible. Nonetheless, it is quite possible that they play a certain role in the complete picture, e.g., as the source of new neutrino states, as some element in addition to the "seesaw - GUT" scenario, or as some element of physics at the deeper level or above the GUT scale. Flavor structure can appear as a result of certain compactification of extra dimensions.

In this review we present detailed discussion of these statements and outline arguments in favor of this possibility. The review is organized as follows. In sect. 2 we introduce main notions on neutrino mass and mixing, and summarize available experimental results. We proceed with the bottom-up approach in the following three sections. In sect 3. we analyze results on neutrino mixings and mass matrix. We address the question of particular "neutrino symmetries" in sect 4. and in sect. 5 we consider possible relations between quarks and leptons. In the rest of the paper the top-down approach is presented: we study properties of the seesaw mechanism in sect 6. Neutrino masses in the grand unified theories are discussed in sect. 7. We consider achievements and limitations of this picture. Need for flavor symmetries in addition to GUTs is described. We also discuss alternative mechanisms of neutrino mass generation and the alternative scenarios of big picture. In sec. 8, we discuss sterile neutrinos, their implications and possible theories. Sec. 9 contains the conclusion.

II. MASSES, FLAVORS AND MIXING

A. Neutrinos and Standard model

According to the standard model (SM), the left handed neutrinos form the electroweak doublets L with charged leptons, have zero electric charge and no color. The right handed components, ν_R, are not included essentially by choice.

The masslessness of the neutrinos at the tree level in this model owes its origin to the fact that there are no right handed neutrinos. This result holds not only to all orders in perturbation theory but also when nonperturbative effects are taken into account due to the existence of an exact B-L (baryon minus lepton number) symmetry in the model even though

B+L is violated by weak sphaleron configurations. It would therefore appear that nonzero neutrino masses must somehow be connected to the existence of right handed neutrinos and/or to breaking of B-L symmetry both of which imply new physics beyond the standard model.

Zero conserved charges (color and electric) distinguish neutrinos from other fermions of the standard model. This difference leads to several new possibilities for neutrino masses all of which involve new physics:

(i) The neutrino masses could be the Majorana type thereby breaking L by two units.

(ii) Neutrinos have the possibility to mix with singlets of SM symmetry group, in particular, singlet fermions in extra dimensions.

The main question is whether these features are enough to explain all salient properties of the neutrino masses and mixing observed in experiment.

There is one way that the neutrino mass can be generated even if the SM particles are the only light degrees of freedom. This requires that one should abandon the condition of explicit renormalizability of the theory. Indeed, the non-renormalizable operator [2]

$$\frac{\lambda_{ij}}{M}(L_i H)^T (L_j H), \quad i, j = e, \mu, \tau, \tag{1}$$

where H is the Higgs doublet λ_{ij} are the dimensionless couplings and M is the cut-off scale, after the electroweak symmetry breaking generates the Majorana neutrino masses

$$m_{ij} = \frac{\lambda_{ij}\langle H\rangle^2}{M}. \tag{2}$$

The operator breaks L and $(B-L)$ quantum numbers. One may think that this operator is generated by some gravitational - Planck scale effects, so that $M \sim M_{Pl}$ and $\lambda_{ij} \sim 1$ [3]. In this case however $m_{ij} \sim 10^{-5}$ eV are too small to explain the observed masses (though such a contribution can still produce some subleading features [4]). Therefore new scales of physics below M_{Pl} must exist to give the desired mass to neutrinos. The operator eq.(1) can appear after integrating out some new heavy degrees of freedom with masses $M \ll M_{Pl}$.

Another important conclusion from this consideration is that the neutrinos can get relevant contributions to masses from all possible energy/mass scales M from ~ 1 eV to the Planck scale. If two or more different contributions (from different scales and different physics) contribute to the mass substantially, interpretation of results can be extremely difficult.

B. Right handed neutrinos, neutrino mass and seesaw.

Let us consider possible extensions of the standard model which can lead to non-zero neutrino masses.

1). If the right handed neutrinos exist (we will consider the conceptual implications of their existence in sect. 6.2) one can introduce the Yukawa coupling

$$Y_\nu \bar{L} H \nu_R + h.c. \tag{3}$$

which leads after the electroweak (EW) symmetry breaking to the Dirac neutrino mass

$$m_D = Y_\nu \langle H \rangle. \tag{4}$$

The observed neutrino masses would require $Y_\nu \leq 10^{-13} - 10^{-12}$. If ν_R is the same type of field as RH components of other fermions, such a smallness looks rather unnatural. However, if the Dirac mass is formed by coupling with some new singlet fermion, S, beyond the usual fermion family structure, possible new symmetries associated to S or/and ν_L can suppress h_ν. In this case h_ν appears as the effective coupling: $h_\nu \sim (v_S/M)^n$, where $v_S \ll M$ are the scales of some new interactions and new symmetry breaking, and n some integer determined by the charges of the fields.

2). The right handed neutrinos are allowed to have Majorana masses

$$M_R \nu_R^T C^{-1} \nu_R + h.c., \tag{5}$$

where C is the Dirac charge conjugation matrix. Since ν_R are singlets under the SM gauge group the M_R can appear as a bare mass term in the Lagrangian or be generated by interactions with singlet scalar field σ (so that $M_R \to f_\sigma \sigma$ in eq.(5)):

$$M_R = f_\sigma \langle \sigma \rangle, \tag{6}$$

where $\langle \sigma \rangle$ is the VEV of σ. The latter possibility is realized if, *e.g.*, ν_R is a component of a multiplet of an extended gauge group.

3). The left handed neutrinos can also acquire the Majorana masses m_L. The corresponding mass terms have the weak isospin $I = 1$ and violate lepton number by 2 units. So they can be generated either via the non-renormalizable operators eq.(1) with two Higgs doublets or/and due to coupling with the Higgs triplet Δ:

$$f_\Delta L^T L \Delta + h.c.. \tag{7}$$

The non-zero VEV of Δ then gives $m_L = f_\Delta \langle \Delta \rangle$.

In the case of 3 neutrino species the mass parameters m_D, M_R and m_L should be considered as 3×3 (in general non-diagonal) matrices. In general all these mass terms are present. Introducing the charge conjugate left handed component $N_L \equiv (\nu_R)^C$ we can write the general mass matrix in the basis (ν_L, N_L), as

$$M_\nu = \begin{pmatrix} m_L & m_D^T \\ m_D & M_R \end{pmatrix}. \tag{8}$$

The eigenstates of this matrix are the Majorana neutrinos with different Majorana masses. The Dirac mass term mixes the active neutrinos with the sterile singlet states N_L.

The matrix has several important limits.

Suppose $m_L = 0$. The Yukawa couplings Y_ν eq.(3) are expected to be of same order as the charged fermion couplings. Since the N_L's are singlets under the SM gauge group, their Majorana masses unlike the masses of the charged fermions, are not constrained by the gauge symmetry and can therefore be arbitrarily large, *i.e.* $M_R \gg m_D$. In this case the diagonalization of the mass matrix eq.(8), leads to an approximate form for the mass matrix for the light neutrinos m_ν as follows:

$$\mathcal{M}_\nu = -\, m_D^T M_R^{-1} m_D. \tag{9}$$

Since as already noted M_R can be much larger than m_D one finds that $m_\nu \ll m_{e,u,d}$ very naturally as is observed. This is known as the seesaw (type-I) mechanism [5] and it provides a natural explanation of why neutrino masses are small.

If elements of the matrix m_L are non-zero but much smaller than the other elements of M_ν, we can write the resulting light neutrino mass matrix in the form

$$\mathcal{M}_\nu = m_L - m_D^T M_R^{-1} m_D. \tag{10}$$

We will refer to this as to mixed seesaw [6, 7] and when the first term dominates, we will call it type II seesaw.

In the matrix eq.(8), it may turn out that the elements of both m_L and M_R have magnitudes which are much smaller than those of m_D. In this case, the neutrinos will predominantly be Dirac type with small admixture of Majorana mass. We will call this case pseudo-Dirac, as noted above [8].

The main question that we will discuss subsequently is whether the see-saw mechanism is enough to explain all the features of neutrino mass and mixing.

C. Flavors and mixing

The electron, muon and tau neutrinos, ν_e, ν_μ, ν_τ, - states produced in association with definite charged leptons: electron, muon and tau correspondingly are called the flavor neutrino states. To give an example, the neutrinos emitted in weak processes such as the beta decay or pion decay with electron or muon are called the electron or muon neutrinos. In the detection process, it is the flavor eigenstate that are picked out since the detection devices are sensitive to charged lepton flavors such as (e, μ, τ). In the SM neutrino flavor states ν_e, ν_μ, ν_τ are the states which form the weak doublets (or weak charged currents) with charged lepton states of definite mass:

$$J^\mu = \bar{l}\gamma^\mu(1-\gamma_5)\nu_l, \quad l = e, \mu, \tau. \tag{11}$$

Notice that phenomenologically defined flavor states, as states produced in certain weak processes, may not coincide precisely with theoretical flavor states (the states from certain weak doublets). The difference can appear due to neutrino mixing with heavy neutral leptons which can not be produced in the low energy weak processes due to kinematics. In fact, this situation is realized in the seesaw mechanism. However the admixture and difference of the states is negligible.

Flavor mixing means that the flavor neutrino states ν_α $(\alpha = e, \mu, \tau)$ do not coincide with neutrino mass eigenstates ν_i $(i = 1, 2, 3)$. That is, the weak charged current processes mix neutrino mass states: the electron, muon and tau neutrinos have no definite masses but turn out to be the coherent combinations of the mass states.

Relation between the flavor $\nu_f \equiv (\nu_e, \nu_\mu, \nu_\tau)$ states and the mass states $\nu \equiv (\nu_1, \nu_2, \nu_3)$ can be written as

$$\nu_f = U_{PMNS}\nu, \tag{12}$$

where U_{PMNS} is 3×3 unitary matrix called the Pontecorvo - Maki - Nakagawa - Sakata lepton mixing matrix [9, 10].

The states $\nu_f \equiv (\nu_e, \nu_\mu, \nu_\tau)$ form the flavor basis. The charged weak currents connect them with charged leptons of definite mass $l \equiv e, \mu, \tau$. Inserting eq. (12) into eq.(11) we can write the weak charged currents as

$$J^\mu = \bar{l}\gamma^\mu(1-\gamma_5)U_{PMNS}\nu. \tag{13}$$

So the lepton mixing matrix connects the neutrino mass eigenstates and charge lepton mass eigenstates in the weak charged currents.

Neutrino mass states are the eigenstates of the Hamiltonian in vacuum and we call mixing eq.(12) the vacuum mixing. Vacuum mixing is generated by the non-diagonal mass matrices.

Apart from the flavor ν_f and mass bases let us introduce the *"symmetry" basis* $(\tilde{\nu}, \tilde{l})$ - the basis in which the underlying theory of the fermion masses is presumably formulated. This can be some flavor symmetry, or GUT, or some dynamical principle, or selection rule originating from string theory. *A priori* we do not know this basis, and in fact, its identification is one of the key problems in the bottom-up approach.

Probably in the symmetry basis both the charged lepton and the neutrino mass matrices are non-diagonal (though the models exist in which the symmetry basis coincides with the flavor basis). Then the mass terms of the Largangian can be written as

$$\mathcal{L}_m \;=\; \tilde{\nu}_L^T C^{-1} \mathcal{M}_\nu \tilde{\nu}_L + \bar{\tilde{l}}_L M_\ell \tilde{l}_R + h.c.. \qquad (14)$$

We have assumed that neutrinos are Majorana particles. (Notice that in the flavor basis (ν_l, l) the mass matrix of charge leptons is diagonal, therefore existence of mixing implies that the mass matrix of neutrinos should be non-diagonal.)

We diagonalize matrices in eq.(14) as

$$U_\nu^T \mathcal{M}_\nu U_\nu = M_\nu^d, \qquad U_\ell M_\ell V_\ell^\dagger = M_\ell^d, \qquad (15)$$

where $M_\nu^d \equiv diag(m_1, m_2, m_3)$ and $M_\ell^d \equiv diag(m_e, m_\mu, m_\tau$ are the diagonal matrices and the rotation matrices, U_ν, U_ℓ, V_ℓ, connect the symmetry states with the mass eigenstates: $\tilde{\nu} = U_\nu \nu$, $\tilde{l}_L = U_l l_L$, $\tilde{l}_R = V_l l_R$. Plugging these relations into the charged current we obtain

$$J^\mu = \bar{\tilde{l}} \gamma^\mu (1 - \gamma_5) \tilde{\nu} = \bar{l} \gamma^\mu (1 - \gamma_5) U_l^\dagger U_\nu \nu. \qquad (16)$$

So, the physical neutrino mixing matrix is then given by

$$U_{PMNS} \;=\; U_\ell^\dagger U_\nu. \qquad (17)$$

It is convenient to parameterize the mixing matrix as

$$U_{PMNS} = U_{23}(\theta_{23}) U_{13}(\theta_{13}, \delta) U_{12}(\theta_{12}) I_\phi, \qquad (18)$$

where U_{ij} matrices of rotations in the ij plane by angle θ_{ij}; δ is the Dirac CP-violating phase attached to 1-3 rotation. In the case of Majorana neutrinos sometimes the mixing matrix is defined as $U'_{PMNS} = U_{PMNS} I_\phi$, where $I_\phi \equiv diag(1, e^{i\phi_1}, e^{i\phi_2})$ is the diagonal matrix of the Majorana CP violating phases. The latter can be included into the mass eigenvalues which can be considered then as the complex parameters.

Notice that the above parameterization allows us to connect immediately the rotation angles with physical observables in the first approximation: $\theta_{23} = \theta_{atm}$, is the angle measured in the atmospheric neutrino oscillations; $\theta_{12} = \theta_{sol}$ is the angle determined from solar neutrino studies, and $\theta_{13} = \theta_{CHOOZ}$ is the angle restricted by the reactor experiment CHOOZ.

D. Experimental results and global fits

We will use the results of the global analysis of the neutrino data published till the end of 2005. It was assumed in the analysis that (i) there are only three mixed active neutrinos; (ii) CPT is conserved, so that masses and mixing angles in the neutrino and antineutrino channel coincide; (iii) neutrino masses and mixings have pure "vacuum origin": that is, due to the interaction with Higgs field(s) which develop the VEV at the scale which is much larger than neutrino mass. We comment later on possible changes when some of these assumptions are abandoned.

The parameter space includes the oscillation parameters: mass squared differences $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$, mixing angles θ as well as the Dirac CP-violating phase δ; non-oscillation parameters are the absolute mass scale which can be identified with the mass of the heaviest neutrino and two Majorana CP-violating phases.

The experimental results used in the analysis can be split into three sectors:

1). Solar neutrinos [11] and the reactor experiment KamLAND [12] are mainly sensitive to Δm_{21}^2 and θ_{12} (solar sector). The 1-3 mixing, if not zero, may give subleading effect.

2). Atmospheric neutrino studies [13] and K2K accelerator experiment [14] are sensitive to Δm_{23}^2 and θ_{23} (atmospheric sector). The solar parameters Δm_{21}^2 and θ_{12} give small sub-leading effects. Also sub-leading effect can be due to nonzero θ_{13}.

3). CHOOZ experiment [15] gives the bound θ_{13} as function of Δm_{31}^2.

The physical effects involved in the interpretation are

- Vacuum oscillations [9, 10, 16] (atmospheric neutrinos - main mode, K2K, CHOOZ);

- MSW effect - the adiabatic conversion [17, 18] (conversion of solar neutrinos in the matter of the Sun; at low energies solar neutrinos undergo the averaged vacuum oscillation with small matter effect);

- Oscillations in matter (oscillations solar and atmospheric neutrinos in the matter of the Earth). These oscillations produce sub-leading effects and have not yet been established at the statistically significant level.

Let us consider the results of global analysis from [19, 20].

1). The mass split responsible for the dominant mode of the atmospheric neutrino oscillations equals

$$|\Delta m_{32}^2| = (2.4 \pm 0.3) \cdot 10^{-3} \text{ eV}^2, \quad (1\sigma). \tag{19}$$

The sign of the mass split determines the type of mass hierarchy: normal $\Delta m_{32}^2 > 0$, or inverted $\Delta m_{32}^2 < 0$, and it is not identified yet. The result eq.(19) allows us to get a lower bound on the heaviest neutrino mass:

$$m_h \geq \sqrt{\Delta m_{13}^2} > 0.04 \text{ eV}, \ (2\sigma), \tag{20}$$

where $m_h = m_3$ for the normal mass hierarchy, and $m_h = m_1 \approx m_2$ for the inverted hierarchy.

Much smaller mass squared split drives the solar neutrino conversion and oscillations detected by KamLAND

$$\Delta m_{12}^2 = (7.9 \pm 0.4) \cdot 10^{-5} \ \text{ eV}^2, \quad (1\sigma). \tag{21}$$

In the case of the hierarchical mass spectrum that would correspond to $m_2 \sim 0.009$ eV. The ratio of the solar and atmospheric neutrino mass scales,

$$r_\Delta \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2} = 0.033 \pm 0.004 \tag{22}$$

gives the lower bound on the corresponding mass hierarchy

$$\frac{m_2}{m_3} \geq \sqrt{r_\Delta} = 0.18 \pm 0.01 \tag{23}$$

(if the hierarchy is normal).

In figs. 1, 2, 3 we summarize results of determination of the mixing angles obtained by different groups. We show also some theoretical benchmarks which will be discussed later.

The best fit value and 1σ error equal

$$\theta_{12} = 33.9^\circ \pm 1.6^\circ, \tag{24}$$

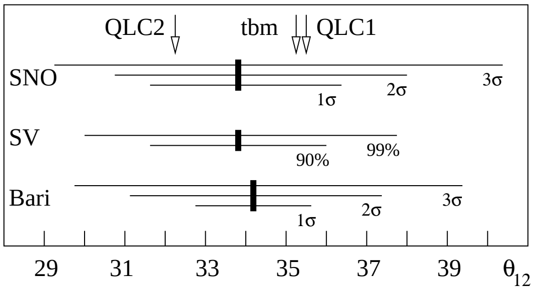

FIG. 1: The best fit values and the allowed regions of lepton mixing angle θ_{12} at different confidence levels determined by different groups. From SNO data (from ref.[11]), SV [19] and Bari [20]. Shown are predictions from QLC and tri-bimaximal mixing (see secs. 4,5).

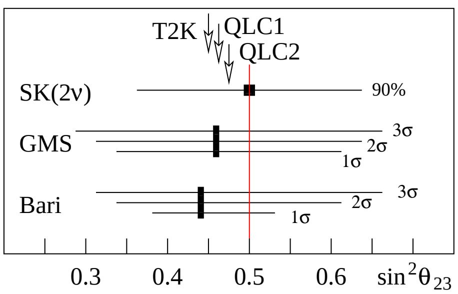

FIG. 2: The best fit values and the allowed regions of $\sin^2\theta_{23}$ at different confidence levels determined by different groups: SK [13], GMS [23], Bari [20]. Shown are expectations from QLC (sec. 5) and sensitivity limit of T2K experiment [21].

or $\sin^2\theta_{12} = 0.315 + 0.028/-0.025$. The central value deviates from maximal mixing, $\sin^2\theta_{12} = 0.5$, by about 6σ.

2). The 2-3 mixing is in agreement with maximal one $\theta_{23} = \pi/4$ (fig. 2). A shift of the best fit point from $\pi/4$ to smaller angles appears when the effect of 1-2 mass split and mixing is included in the analysis [20, 23]. According to [23] $\sin^2\theta_{23} = 0.47$ and slightly larger shift, $\sin^2\theta_{23} = 0.44$, follows from the analysis [20]. So, the deviation from maximal mixing can be quantified as

$$D_{23} \equiv 0.5 - \sin^2\theta_{23} \sim 0.03 - 0.06. \tag{25}$$

The shift is related to an excess of the so called e-like atmospheric neutrino events in the sub-

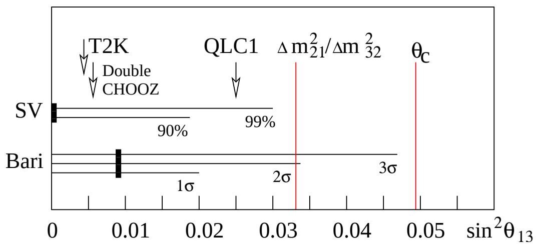

FIG. 3: The best fit values and allowed regions of $\sin^2\theta_{13}$ at different confidence levels determined by different groups: SV [19] and Bari [20]. Shown are also some theoretical predictions and sensitivity limits of Double CHOOZ [22] and T2K [21].

GeV range. The excess can be explained by the oscillations driven by the solar parameters and it is proportional to the deviation $\Delta N_e/N_e \propto D_{23}$ [24]. The experimental errors still allow substantial deviation from maximal mixing:

$$D_{23}/\sin^2\theta_{23} \sim 0.4 \quad (2\sigma). \tag{26}$$

3). Results on the 1-3 mixing are consistent with zero θ_{13} (fig. 3). Small non-zero best fit value of $\sin^2\theta_{13}$ from the analysis [20] is related to the angular distribution of the multi-GeV e-like events measured by SuperKamiokande [13]. The most conservative 3σ bound is [20]

$$\sin^2\theta_{13} < 0.048, \quad (3\sigma). \tag{27}$$

In the first approximations the pattern of lepton mixing has been established. There are two large mixings: the 2-3 one is consistent with maximal, 1-2 is large but not maximal and 1-3 mixing is small and is consistent with zero.

Further precise measurements of the mixing angles and in particular, searches for the deviations of 1- 3 mixing from zero and 2-3 mixing - from maximal, is crucial for understanding the underlying physics. The figures show the accuracy required to make important theoretical conclusions and the potential of next generation experiments.

Information on non-oscillation parameters has been obtained from the direct kinematical measurements, neutrinoless double beta decay and cosmology.

The effective Majorana mass of the electron neutrino - the ee-element of the neutrino mass matrix, m_{ee}, determines the rate of the neutrinoless double beta decay: $T(2\beta0\nu) \sim m_{ee}^{-2}$. In

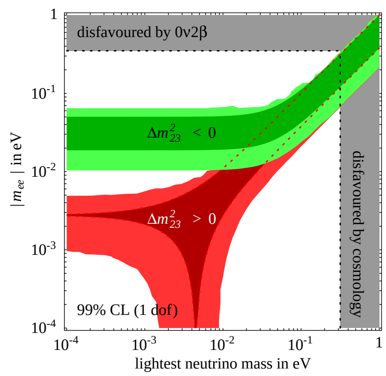

FIG. 4: The 90% CL range for m_{ee} as a function of the lightest neutrino mass for the normal ($\Delta m^2_{23} > 0$) and inverted ($\Delta m^2_{23} < 0$) mass hierarchies. The darker regions show how the allowed range for the present best-fit values of the parameters with negligible errors; from [19].

terms of masses and mixing parameters it can be written as

$$m_{ee} = |\sum_k U^2_{ek} m_k e^{i\phi(k)}| = \left|\sum_k U^2_{ek}\sqrt{m^2_L + \Delta m^2_{kL}}\, e^{i\phi(k)}\right|, \tag{28}$$

where $\phi(k)$ is the phase of the k eigenvalue, U_{ek} its admixture in ν_e, and m_L is the lightest neutrino mass. Fig. 4 from [19] summarizes the present knowledge of the absolute mass scale. Shown are the regions allowed by oscillation results in the plane of m_{ee} and m_1 - the mass of lightest neutrino probed by the direct kinematical methods and cosmology. The two bends correspond to the normal and inverted mass hierarchies. For a given m_1 the range of m_{ee} is determined by variations of the Majorana phases ϕ and uncertainties in the oscillation parameters.

The best present bound on m_{ee} is given by the Heidelberg-Moscow experiment: $m_{ee} < (0.35 - 0.50)$ eV [25]. Part of the collaboration claims an evidence for a positive signal [26] which would correspond to $m_{ee} \sim 0.4$ eV. If this positive signal Heidelberg-Moscow result is confirmed and if it is due to exchange of the light Majorana neutrinos, the neutrino mass spectrum should be strongly degenerate: $m_1 \approx m_2 \approx m_3 \equiv m_0$ [27].

The cosmological observations put the bound on sum of neutrino masses
$\sum_{i=1}^3 m_i < 0.42$ eV (95% $C.L.$) [28] which corresponds to $m_0 < 0.13$ eV in the case of degenerate spectrum. Even stronger bound, $\sum_{i=1}^3 m_i < 0.42$ eV (95% C.L.) has been established recently [29].

Combining the cosmological and oscillation (20) bounds, we conclude that at least one neutrino mass should be in the interval

$$m \sim (0.04 - 0.10) \text{ eV} \quad (95\% \ \text{ C.L.}). \tag{29}$$

Direct kinematic measurements give weaker bound $m < 2.0 - 2.2$ eV [30]. The planned experiment KATRIN [31] is expected to improve this limit down to ~ 0.2 eV.

How robust are these results? Can we expect some substantial change in this picture in future? There are three types of effects (in fact, related to lifting of assumptions made in the analysis) which can influence interpretation of the present neutrino results:

1). Possible existence of new neutrino states - sterile neutrinos. If these states are light they can directly (dynamically) influence the observed effects used to determine neutrino parameters.

2). Possible presence of the non-standard neutrino interactions can change values of the extracted neutrino parameters.

3). Interactions with light scalar fields [32] can produce the soft "neutrino masses" which depend on properties of medium. These masses may also change with time and be related to dark energy in the universe [33].

At present, however there is no well established results which would indicate deviation from the "standard" 3ν mixing and standard matter interactions.

There are various tests of validity of the standard picture and theory of neutrino conversion. Different sets of the data confirm each other. Consistent interpretation of whole bulk of various data in terms of the vacuum masses and mixing provides us with further confidence. The fit of the data is not improved with inclusion of new states and the non-standard interactions and, if exist, they may produce sub-leading effects only. One can perform, *e.g.*, the global fit of neutrino data considering normalization of the matter potential as free parameter. According to [20] the best fit value of the potential is close to the standard one. In this way one tests not only the validity of the refraction theory for neutrinos but also consistency of the whole picture.

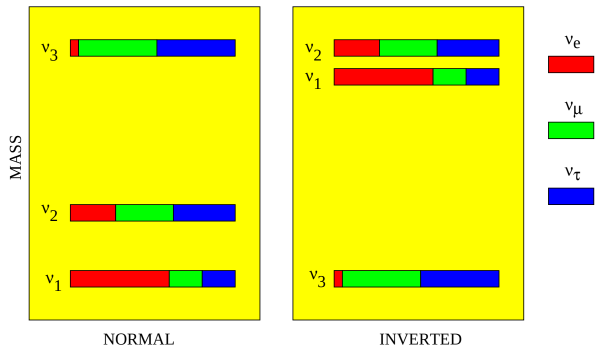

FIG. 5: Neutrino mass and flavor spectra for the normal (left) and inverted (right) mass hierarchies. The distribution of flavors (colored parts of boxes) in the mass eigenstates corresponds to the best-fit values of mixing parameters and $\sin^2\theta_{13} = 0.05$.

E. Mass and flavor spectrum

Information obtained from the oscillation experiments allows us to partially reconstruct the neutrino mass and flavor spectrum (Fig. 5).

Determination of unknowns comprises the program of future phenomenological and experimental studies. Those include (i) admixture of ν_e in ν_3 described by U_{e3}; (ii) type of mass spectrum: hierarchical; non-hierarchical with certain ordering; degenerate, which is related to the value of the absolute mass scale, m_1; (iii) type of mass hierarchy (ordering): normal, inverted; (iv) CP-violating phase δ.

There are some weak indications in favor of normal mass hierarchy from supernova SN1987A data. However in view of small statistics and uncertainties in the original fluxes it is not possible to make a firm statement.

As is clear from the fig. 4, future high sensitivity measurements of the effective mass m_{ee} can allow one to establish the hierarchy: The bound $m_{ee} < 0.01$ eV will exclude the inverted mass hierarchy and also degenerate mass spectrum. Future detection of the galactic supernova can also contribute to the determination of the type of mass hierarchy and 1-3 mixing [34].

F. Towards the underlying physics

What is behind all these observations? To uncover the underlying physics two general strategies are invoked:

(1) Bottom-Up approach. This is essentially an attempt to uncover the underlying physics starting from observations. The strategy is to (i) reconstruct the neutrino mass matrix in the flavor basis using the available information on masses and mixings (Δm^2_{ij}, θ_{ij}, m_{ee}); (ii) take into account the renormalization group effect and obtain the mass matrix at the scale of new physics, (iii) search for the "symmetry" basis in which flavor or some other symmetry is realized; (iv) identify the symmetry and mechanism of symmetry violation, if needed, as well as the underlying dynamics. This is our standard way to understand things but it is not excluded that explanation will require something new.

(2) Top-Down approach: Here one starts with a general unified theory framework, be it a grand unified theory, TeV scale theory or extra dimension theory which has motivation outside neutrino physics - and use it to make predictions for neutrinos. That is, go from big picture to the observed properties of neutrinos.

At some point these two approaches should merge. Both approaches are needed: it seems difficult to uncover the underlying picture just moving from observations and some *a priori* ideas (or context) are needed to relate neutrinos with other physics. This opens a possibility to use results from other areas (e.g. collider experiments). So we need the big picture. On the other hand, working solely within the "big picture", one can miss some important elements which is where the bottom up approach can help.

The results of the bottom-up approach are considered in sect. 3 - 5. The top-down approach is developed in sect 6 -7.

III. MIXING AND MASS MATRIX OF NEUTRINOS

Analyzing results on neutrino mixing and mass matrix one can (i) search for some particular features in the data: empirical relations, equalities, hierarchies of elements, zeros, *etc.*; (ii) identify possible dominant structures in the mixing and mass matrices (the idea being that matrices can have a structure as "lowest order plus small corrections" which in turn can correspond to some dominant mechanism plus sub-dominant effects); (iii) search for simple parameterization in terms of small number of parameters; (iv) present matrices in terms of powers of some small quantity, *etc.*. All these may give some hint of the underlying physics.

A. Properties of neutrino mixing matrix

Let us first to analyze the mixing matrix. Maximal 2-3 mixing, large 1-2 mixing and small 1-3 mixing indicate that the following matrices can play some important role (*e.g.*, be the lowest order mixing matrices).

1). The bi-maximal mixing matrix [35]:

$$U_{bm} = U_{23}^m U_{12}^m, \tag{30}$$

where U_{23}^m and U_{12}^m are the matrices of the maximal ($\pi/4$) rotations in the 2-3 and 1-2 subspaces correspondingly. Explicitly, we have

$$U_{bm} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \end{pmatrix}. \tag{31}$$

Identification $U_{PMNS} = U_{bm}$ is not possible due to strong $\sim 6\sigma$ deviation of the 1-2 mixing from maximal. However, U_{bm} can play the role of a dominant structure. In the latter case, the correction can originate from the charged lepton sector (mass matrix), so that $U_{PMNS} = U'U_{bm}$. Suppose $U' \approx U_{12}(\alpha)$ in analogy with quark mixing. Then U' generates simultaneously deviation of the 1-2 mixing from maximal and non-zero 1-3 mixing which are related as

$$\sin\theta_{13} = \frac{\sin\alpha}{\sqrt{2}}$$

and

$$\sin\theta_{23} \simeq \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2}\tan^2\theta_{13}\right)$$

or $D_{23} \approx 0.5\tan^2\theta_{13}$. Confirmation of this equality will be very suggestive.

2). The tri-bimaximal mixing matrix [36]

$$U_{tbm} = U_{23}^m U_{12}(\theta_{12}), \quad \sin^2\theta_{12} = 1/3, \tag{32}$$

or explicitly,

$$U_{tbm} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ 1 & -\sqrt{2} & \sqrt{3} \end{pmatrix} \tag{33}$$

is in good agreement with data including 1-2 mixing. Here ν_2 is tri-maximally mixed: in the second column three flavors mix maximally, whereas ν_3 is bi-maximally mixed. Mixing parameters turn out to be simple numbers like 0, 1/3, 1/2 and can appear as the Clebsch-Gordon coefficients.

The bi-maximal and tri-bimaximal matrices can be considered as matrices in the lowest order of some approximation. Then one can introduce parameters which describe deviations of the true matrix from these lowest order structures [37]. The matrices U_{bm} and U_{tbm} reveal certain symmetries, and then the deviation parameters describe effects of violation of these symmetries.

B. Reconstructing neutrino mass matrix

Mass matrix is probably more fundamental than mass eigenvalues and mixing angles since it combines information about masses and mixings. Dynamics and symmetries can be realized in terms of mass matrices and not its eigenstates and eigenvalues. However, it is possible (and models of this type exist, see below) that symmetry determines immediately the mixings and not masses which are left as free parameters.

As we have mentioned, the first step in the bottom-up approach is the reconstruction of the mass matrix in flavor basis. Notice that in the case of Majorana neutrinos, the elements of this mass matrix are physical parameters: they can be directly measured, *e.g.*, in neutrinoless double beta decay and, in principle, in other similar processes.

The answer to the question what is more fundamental: mass matrices or observables (Δm^2, θ), may depend on the type of mass spectrum. In the case of hierarchical spectrum, the observables are visibly imprinted into the structure of the mass matrix. In contrast, for the quasi-degenerate spectrum they are just very small perturbations of the dominant structure determined by the non-oscillatory parameters: the absolute mass scale and the Majorana CP-violating phases. The oscillation parameters can originate from some small, in particular radiative, corrections.

In the flavor basis the mass matrix of the charged leptons is diagonal and therefore the neutrino mass matrix is diagonalized by $U_\nu = U_{PMNS}$. Consequently, according to eq.(15) the neutrino mass matrix in the flavor basis can be written as

$$\mathcal{M}_\nu = U^*_{PMNS} \mathcal{M}^d_\nu U^\dagger_{PMNS}, \tag{34}$$

where

$$\mathcal{M}^d_\nu \equiv diag(m_1,\ m_2 e^{-2i\phi_2},\ m_3 e^{-2i\phi_3}). \tag{35}$$

where ϕ_i are the Majorana phases and we can take $\phi_1 = 0$. Results of reconstruction show [38] that a large variety of different structures of mass matrices is possible, depending strongly on the unknown m_1, type of mass hierarchy and Majorana phases. The dependence on $\sin\theta_{13}$ and δ is relatively weak. This means huge degeneracy of mass matrices now and perhaps even in the far future since in reality, it is not possible to measure all the parameters including CP-violating phases. Variations of one Majorana phase (even if all other parameters are known) can lead to strong change of the structure. Nature should be very "collaborative" with us to let us know the mass matrix. Or we may uncover some principle which will allow us to predict the mass matrix, which we can then check by certain precision measurements.

C. Extreme cases

To give some idea about various possibilities, we will present simple parameterizations of the neutrino mass matrix in the flavor basis for three extreme cases: normal mass hierarchy, inverted mass hierarchy and degenerate spectrum.

1). Normal mass hierarchy ($m_1 \ll m_2 \ll m_3$). The mass matrix indicated by data can be parameterized as

$$\mathcal{M}_\nu \;=\; \frac{\sqrt{\Delta m^2_A}}{2} \begin{pmatrix} d\epsilon & b\epsilon & a\epsilon \\ b\epsilon & 1 + c\epsilon & -1 \\ a\epsilon & -1 & 1+\epsilon \end{pmatrix}, \tag{36}$$

where a, b, c, d are complex parameters of the order 1, and ϵ is essentially the ratio of the solar and the atmospheric mass hierarchies squared: $\epsilon = 2\sqrt{r} F(a,c,b,d);\ F(a,c,b,d) \sim O(1)$. Salient feature of this matrix is the dominant $\mu-\tau$ block. Actually, with the present accuracy of measurements of the parameters it is not excluded that sharp difference between dominant and subdominant elements does not exist and some moderate hierarchy of elements with the unique expansion parameter 0.6 - 0.7 is realized [38].

An important property of the above mass matrix is that in the limit of $a = b$ and $c = 1$, it is symmetric under $\mu - \tau$ interchange and one gets maximal atmospheric angle and zero 1-3 mixing, *i.e.* $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ [39]. This symmetry should of course be approximate since the masses of muon and tau leptons are different. Any resulting $\mu - \tau$ breaking must therefore reflect itself in nonzero θ_{13} and D_{23} with both connected to each other [40]. For example, if $a = b$ and $c \neq 1$, that is, the symmetry is broken in the dominant block, the induced θ_{13} and D_{23} are given by

$$\sin\theta_{13} = -b\epsilon^2(c-1)/4, \qquad D_{23} = \epsilon(c-1)/4, \tag{37}$$

and they are strongly correlated:

$$\tan\theta_{13} = b\epsilon D_{23}. \tag{38}$$

Furthermore, $\theta_{13} \ll D_{23}$. In contrast, if the symmetry is broken in the sub-dominant block only: $a \neq b$ but $c = 1$, the situation is opposite: $\theta_{13} \gg D_{23}$, *i.e.*

$$\sin\theta_{13} \simeq \frac{a-b}{\sqrt{2}}\epsilon, \qquad D_{23} = \frac{b^2-a^2}{8}\epsilon^2. \tag{39}$$

So, measurements of θ_{13} and D_{23} will provide an important probe of the mass matrix structure.

Note that when $a = b = d$ and $c = 1$, we get the tri-bi-maximal mixing pattern.

2). Inverted mass hierarchy ($m_1 \approx m_2 \gg m_3$). The structure of the mass matrix in this case depends strongly on the CP-violation phase. An approximate form of the mass matrix in the case of opposite CP parities of ν_1 and ν_2 is:

$$\mathcal{M}_\nu = \sqrt{\Delta m_A^2}\begin{pmatrix} z\epsilon & c & s \\ c & y\epsilon & d\epsilon \\ s & d\epsilon & x\epsilon \end{pmatrix} \tag{40}$$

where $\epsilon \ll 1$. In the limit of $\epsilon \to 0$, this mass matrix has the symmetry $L_e - L_\mu - L_\tau$ [41]. In the symmetry limit one has $\Delta m_{12}^2 = 0$ and $\theta_{12} = \pi/4$. Furthermore, if an additional $\mu - \tau$ exchange symmetry is imposed on this mass matrix, the atmospheric mixing angle also becomes maximal.

The breaking of $L_e - L_\mu - L_\tau$ symmetry leads to nonzero Δm^2_{12} and deviation from maximality of θ_{12}. It has however been generally hard, though not impossible, to accommodate the observed "large" departure from maximality of θ_{12} using "small" breakings of $L_e - L_\mu - L_\tau$ symmetry. One needs as much as 40% breaking to fit data.

In the case of the same CP-parities of mass eigenstates the mass matrix has completely different form with interchange of dominant - sub-dominant elements in eq.(40). This illustrates strong dependence of the matrix structure on the unknown CP violating phases.

3). Degenerate spectrum: $m_1 \approx m_2 \approx m_3 = m_0$. Here also the structure of mass matrix depends strongly on the CP-violating phases. Two possibilities are particularly interesting:

(i) If the relative phases between the mass eigenvalues is zero ($2\pi k$), the mass matrix is close to the unit matrix

$$\mathcal{M}_\nu = m_0 I + \delta M, \tag{41}$$

where $\delta M \ll m_0$ is the matrix of small corrections.

(ii) In the case of opposite CP-parities of ν_2 and ν_3

$$\mathcal{M}_\nu = m_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \delta M. \tag{42}$$

Both matrices give $m_{ee} = m_0$ and can explain the Heidelberg-Moscow positive result.

Theoretical understanding of such a situation would then require that there must be underlying symmetries that guarantees mass degeneracy. The basic strategy here is to consider symmetries that have a three dimensional representation to which the three lepton doublets of the standard model can be assigned and then design a Higgs sector that will lead to a quasi-degenerate neutrino spectrum. A list of such symmetries includes S_4 [42], $SO(3)$ [43], A_4 [44].

In this connection an interesting possibility is a theory with mixed seesaw, where the type II contribution gives the dominant quasi-degenerate term $m_0 I$ or the first term in eq. (42) with the conventional type I contribution giving mass splittings and mixings : $\delta M = -m_D^T M_R^{-1} m_D$. A very generic way to see how these models could explain observations is as follows: In a quark-lepton unified theory such as SO(10) model, we would expect the

Dirac mass term for the neutrinos to have a hierarchical pattern for its eigenvalues so that roughly speaking, the atmospheric and solar mass differences will be given by

$$\Delta m^2_{13} \sim \frac{m_0 m^2_{D,33}}{M_3}, \quad \Delta m^2_{12} \sim \frac{m_0 m^2_{D,22}}{M_2}$$

respectively, roughly similar to observations.

The matrices eqs.(41,42) open various possibilities to relate the degeneracy of the spectrum with large or maximal mixing. The matrix eq.(42) immediately leads to maximal (2-3) mixing. Nearly maximal mixings are generated by small off-diagonal elements in eq.(41), *etc.*.

D. Mass matrices with texture zeros

Another approach in analyzing possible mass matrices is to see if some elements can be exactly zero or equal each other. This may also uncover dominant structures and certain underlying symmetries. This approach allows one also to reduce the number of free parameters and therefore can lead to certain predictions. Recall that the Majorana mass matrix for three neutrinos has 6 independent elements.

Mass matrices with different numbers of zeros and with zeros in various places of matrix have been considered. Two of the cases discussed widely in the literature are textures with (i) three zeroes and (ii) two zeros. It is easy to convince oneself that the three zeroes cannot be along the off diagonal entries nor can they be in any of the 2×2 submatrices and yet give a fit to already known data. In the former case all mixings vanish and in the latter case, one cannot satisfy the requirement from observation that $\Delta m^2_{12} \ll \Delta m^2_{23}$ if θ_{23} and θ_{12} are large as observed.

The case when all zeros are along the diagonal [45] (or two along diagonal and third is off diagonal) is more subtle since now one can satisfy the requirements of large solar and atmospheric mixings as well as $\Delta m^2_{12} \ll \Delta m^2_{23}$. However in this case, there are only three (real) parameters which can be determined from Δm^2_{12} Δm^2_{23} and θ_{23}. Then one predicts a value for the solar mixing angle, $\sin^2 2\theta_{12} = 1 - r_\Delta/16$, which is incompatible with observations. Thus, neutrino mass matrices with three texture zeros are not viable candidates for neutrino oscillations.

As far as textures with two zeros are concerned, they have five free parameters, four real parameters and a complex phase and are therefore interesting candidates for neutrino mass matrices [46]. These have been analyzed to give their characteristic predictions. There are seven different possibilities (out of fifteen ones) that are currently in accord with data and make predictions for various parameters such as neutrinoless double beta decay and θ_{13}. As an interesting example, consider the matrix

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{pmatrix}, \tag{43}$$

where X indicate non-zero entries. This leads to a hierarchical mass matrix with the prediction for

$$\sin^2\theta_{13} \sim \frac{r_\Delta}{\tan^2\theta_{12} - \cot^2\theta_{12}} \sim 0.01 \tag{44}$$

and zero amplitude for neutrinoless double beta decay.

Another possible texture is

$$\mathcal{M}_\nu = \begin{pmatrix} X & X & 0 \\ X & 0 & X \\ 0 & X & X \end{pmatrix}. \tag{45}$$

This leads to degenerate mass matrix with an effective mass in neutrinoless double beta decay exceeding 0.1 eV.

Such an approach should be taken with some caution: (i) for instance, it is not clear why the zeros appear in the flavor basis? (ii) there is the possibility that interesting symmetries may not correspond to zeros in the flavor basis, and (iii) finally, the zeros may not be exactly zeros, in which case we are unlikely to learn much from these exercises. In any case, such situations should be considered on equal footing with cases where various elements of the neutrino mass matrices are related or simply equal, as in the case of $\mu-\tau$ symmetric models discussed in sec. 4.

E. Anarchy approach

While the limiting examples of mass matrices described above may contain interesting hints of symmetries, it is quite possible that one is far from these cases. In the large part of the parameter space, the mass matrix which explains the data has no clear structure: all the elements are of the same order and could be taken as random numbers. This fact and the existence of two large mixings motivate the anarchy approach [47] where all parameters of the neutrino mass matrix are allowed to take random values. One then calculates the probability that θ_{12}, θ_{23} and θ_{13} satisfy the experimental results. There can be variations to this approach depending on whether the anarchy is assumed to be in the high scale sector of the theory (such as in the right handed neutrino mass matrix in the seesaw models to be described below) or in low scale sector. One feature of these models is that they generally predict "large" θ_{13} (closer to 0.1 or higher) for a large domain of parameters.

One can also consider partial "anarchy" - randomness of the parameters on top of a dominant structure determined by certain symmetry.

Questions for this approach are: which kind of physics leads to the anarchy? Why anarchy manifests for lepton mixings and neutrino masses only? To some extend study of anarchy can be considered as a test of complexity behind neutrino masses and mixings. In fact, it could be that there exist several comparable contributions to the neutrino mass matrix each having a simple structure and obeying certain symmetry but the totality of it giving the anarchy effect.

F. RGE effects

If the underlying theory is formulated at some high energy scale M, *e.g.*, much above the electroweak scale, one needs to use the renormalization group effects to extrapolate from the low energy scale where the mass matrix has been reconstructed to the high scale M. In general, renormalization can change the structure of the Yukawa coupling matrix. So, to uncover the mechanism of mass generation one needs to calculate RGE corrections [48]. In fact, some features of the mass matrix and observables at low scale can be due to renormalization group effect.

Suppose neutrinos are the Majorana particles and their masses are generated at the electroweak scale by the operator (1) (For renormalization of masses of the Dirac neutrinos see [49].) Then in general two RGE effects should be taken into account:

1). Renormalization of the operator in eq.(1) from the low scale up to the scale where it is formed.

2). Renormalization between and above the scale of the operator formation [50]. In general, different terms of the operator in eq.(1) are generated at different scales separated by many orders of magnitude rather than at the unique scale. This happens, *e.g.*, in the case of seesaw type I mechanism with strongly hierarchical masses of the right handed neutrinos: $M_1 \ll M_2 \ll M_3$. The underlying physics is formulated at $M \geq M_3$. In this case one should take into account "threshold effects" - different renormalization group running between the masses (see sect. 6).

In this section we will consider the RGE effects below the scale of formation of operator in eq.(1). In the seesaw version with strong hierarchy of masses that would correspond to $\mu < M_1$, where μ is the running scale.

In the SM as well as MSSM, treatment of the RGE effects on the mass matrix in the flavor basis is simpler than RGE effects on observables - angles or masses. The observables can be found after renormalization performing diagonalization of mass matrix (matrix of the Yukawa couplings).

The RG equation for the effective mass matrix has a very transparent structure [48, 51]

$$\frac{d\mathcal{M}_\nu}{dt} = C_l {Y_l}^\dagger Y_l \mathcal{M}_\nu + \mathcal{M}_\nu C_l {Y_l}^\dagger Y_l + \alpha \mathcal{M}_\nu, \tag{46}$$

where $t \equiv (1/16\pi^2)\log(\mu/\mu_0)$, $C_l = -3/2$ in the SM and $C_l = 1$ in MSSM. The first two flavor dependent terms correspond to the neutrino wave function renormalization due to Yukawa couplings of the charged leptons, the last term is the flavor independent renormalization due to gauge couplings and also Yukawa coupling renormalization of the Higgs field wave function.

1. Renormalization of the neutrino mass matrix.

In the lowest order the gauge couplings produce only the overall renormalization of the mass matrix and do not change its flavor structure. It is not true for threshold corrections since couplings of different RH neutrinos are flavor dependent [50]. In contrast, the Yukawa interactions modify the flavor structure of the mass matrices. In the flavor basis Yukawa corrections do not generate the off-diagonal elements of the charged lepton mass matrix in the SM and MSSM. This matrix remains diagonal and therefore the correction do not change the flavor basis. On the contrary, RGE corrections change structure of the non-diagonal neutrino mass matrix.

To understand the RGE effects we will consider the one loop corrections and neglect all the Yukawa couplings except the tau lepton one: $Y_\tau = m_\tau / v_d$, where v_d is the VEV which generates masses of down fermions. Essentially the RGE effects are reduced to the wave function renormalization and can be written as

$$\mathcal{M}_\nu(\mu) = I_C(\mu) R(\mu) \mathcal{M}_\nu(m_Z) R(\mu). \tag{47}$$

Here I_C is flavor independent renormalization factor and

$$R \approx diag(1, 1, Z_\tau(\mu)), \qquad Z_\tau - 1 = C_l \frac{h_\tau^2}{16\pi^2} \log \frac{\mu}{m_Z}, \tag{48}$$

The size of the effect is different in the SM and MSSM: In SM $v_d = v = 265$ GeV and $h_\tau = 9.5 \times 10^{-3}$ at the EW scale, so the corrections are very small: For $\mu = 10^{10}$ GeV we obtain $(Z_\tau - 1)_{SM} \simeq 10^{-5}$. The effect is strongly enhanced in MSSM with large $\tan\beta$, where $v_d \sim v/\tan\beta$, so that $(Z_\tau - 1)_{MSSM} \approx (Z_\tau - 1)_{SM} \tan^2\beta$. For $\tan\beta = 50$ we obtain $Z_\tau - 1 \sim 0.03$.

Corrections appear as factors multiplying the bare values of the matrix elements. So, the zero elements will remain zeros [52].

This allows us to draw important conclusion: The RGE effects (at least in the SM and MSSM) do not change the structure of the mass matrix significantly. Therefore the mass matrix reconstructed at low energies will have nearly the same form at high scales (before threshold corrections are included). This is not true if some new interactions exist above the EW scale (see *e.g.* [53]).

In contrast, effect of corrections on the observables - angle and mass differences - can be strong for particular forms of the zero order mass matrix.

2. *Renormalization of observables.*

The strongest effect on the observables is in the case of the quasi-degenerate mass spectrum [54, 55, 56] . Indeed, the correction are proportional to the absolute mass scale m_0: $\delta m = m_0(Z_\tau - 1)$. It generates the mass squared difference $\Delta m^2 \approx 2m_0 \delta m = 2m_0^2(Z_\tau - 1)$. Effect increases as square of the overall mass scale. For $m_0 = 0.3$ eV and $(Z_\tau - 1) \sim 10^{-3}$ it can give the solar mass split [56]. Furthermore, the mixing angles depend on mass differences whereas corrections are proportional to the absolute values of the elements so that the relative corrections to the mixing get enhanced [55] by $\Delta\theta \propto m/\Delta m \approx\propto m_0^2/\Delta m^2$. Essentially, the enhancement of mixing occurs when neutrinos become even more degenerate at low energies. In the case of normal hierarchy, however, the effect of the RGE's is small.

Let us summarize possible effects [54, 55, 56, 57, 58, 59].

1). The angle θ_{12} can undergo the strongest renormalization since it is associated to the smallest mass split. Some important dependences and results can be traced from the approximate analytical expression for running [57]:

$$\dot{\theta}_{12} \approx -A \sin 2\theta_{12} \sin^2\theta_{23} \frac{|m_1 + m_2 e^{\phi_{12}}|^2}{\Delta m_{21}^2} + \mathcal{O}(\theta_{13}), \tag{49}$$

where $A \equiv C_l Y_\tau^2/32\pi^2$ and $\phi_{12} \equiv \phi_2 - \phi_1$. Notice that other parameters in this formula, and especially Δm_{12}^2, also run and their dependence on renormalization scale μ should be taken into account. As a result the dependence of θ_{12} on the $log\mu$ is nonlinear.

According to eq.(49) for $\theta_{13} = 0$, with increase of μ the angle θ_{12} decreases in the MSSM and increases in SM. For the degenerate spectrum the enhancement factor can reach $4m_0^2/\Delta m_{12}^2$. Maximal enhancement corresponds to zero relative phase, $\phi_{12} = 0$, and running is suppressed for the opposite phases.

For the degenerate spectrum the angle ϕ_{12} can run practically to zero. This means that the large 1-2 mixing at low energies may have the radiative origin being small at *e.g.* the GUT scale [54, 55]. Another interesting possibility is that at M_{GUT} $\theta_{12} = \theta_C$ - equality of the quark and leptonic 1-2 mixings is realized. So, the difference of quark and lepton mixings is related via the RGE running to degeneracy of the neutrino spectrum.

In MSSM The angle θ_{12} can increase with μ due to non-zero θ_{13} and $\phi_{12} = \pi$, where the last equality ensures that the effect of the first term is suppressed.

2). Evolution of 1-3 mixing associated to the larger mass split is weaker and nearly linear in $\log\mu$. It can be approximated as

$$\begin{aligned}\dot{\theta}_{13} \approx A \sin 2\theta_{12} \sin 2\theta_{23} \frac{m_3}{\Delta m_{13}^2} [m_1 \cos(\phi_1 - \delta) - m_2 \cos(\phi_2 - \delta) \\ - r_\Delta m_3 \cos\delta] + \mathcal{O}(\theta_{13}).\end{aligned} \tag{50}$$

The enhancement factor for the degenerate spectrum is $m_0^2/\Delta m_{13}^2$ and the strongest evolution is when phases ϕ_1 and ϕ_2 are different. For equal phases running is suppressed by an additional factor r_Δ. In the case of normal mass hierarchy θ_{13} decreases with $\log\mu$.

The main term in eq.(50) does not depend on θ_{13} so it evolves to nonzero value even if $\theta_{13} = 0$ at some high energy scale. At low scales non-zero θ_{13} may have purely radiative origin. For instance, one can get $\theta_{13} \sim 8^\circ$ at the level of present experimental bound if it is zero at M_{GUT} [57].

If hierarchy is inverted the θ_{13} increases with μ and at the GUT scale it could be $> \theta_C$. The RGE effect is strongly suppresses when m_3 is small.

3). Running of the 2-3 mixing can be described approximately by [57]

$$\begin{aligned} \dot{\theta}_{23} \approx -A \sin 2\theta_{23} \frac{1}{\Delta m_{23}^2} [&\sin^2 \theta_{12} |m_1 e^{\phi_1} + m_3|^2 \\ &+ \cos^2 \theta_{12} |m_2 e^{\phi_2} + m_3|^2] + \mathcal{O}(\theta_{13}). \end{aligned} \tag{51}$$

As in the previous case the enhancement factor for the degenerate spectrum is $m_0^2/\Delta m_{23}^2$ and running is suppressed if $\phi_1 = \phi_2 = \pi$. In MSSM with increase of $\log \mu$ the angle θ_{23} decreases and can be as small as $(20 - 30)^\circ$ at the GUT scale. So, one can obtain the *radiative enhancement* of the mixing [48, 54, 55]: θ_{23} is small (similar to θ_C) at high energies and it reaches $\sim 45^\circ$ at low energies. Here again the large lepton mixing is related to the neutrino mass degeneracy.

The RGE should lead to deviation of the 2-3 mixing from maximal when running to small scales if it is maximal at high scale and we have $D_{23} = \frac{1}{2}(Z_\tau - 1)$ [59], though for normal hierarchy the deviation is below 1°.

Finally let us consider the renormalization of the 1-2 mass split:

$$\Delta \dot{m}_{12}^2 \approx \alpha \Delta m_{12}^2 - 4A[2 \sin^2 \theta_{23} (m_2^2 \cos^2 \theta_{12} - m_1^2 \sin^2 \theta_{12}) + \mathcal{O}(\theta_{13})], \tag{52}$$

where the first term is the overall renormalization of all masses due to the gauge radiative corrections and also renormalization of the Higgs boson wave function. The second term can dominate for the degenerate spectrum. Depending on parameters the renormalization can enhance splitting up to the atmospheric one or suppress it down to zero. So, zero split at some high scales and radiative origin of the 1-2 split at low scales can be realized [56]. Another possibility is that at high scales all the mass splits are of the same order and the hierarchy of splits at low scales is produced by the radiative corrections in the case of the degenerate spectrum.

As is clear from our discussion the role of the RGE effects depends on a number of unknowns: possible extensions of the SM like two higgs doublet model, MSSM, (ii) on value of $\tan\beta$, (iii) on type of spectrum (degenerate, hierarchical), (iv) on CP violating phases. Depending on these unknowns the effects can vary from negligible to dominant, thus explaining main features of the neutrino mass spectrum and mixing. Apparently many uncertainties related to RGE effects will disappear if it turns out that the neutrino mass spectrum is hierarchical. Even in this case the corrections can be larger than accuracy of future measurements of the neutrino parameters.

Strong effects are expected also from the "threshold" corrections [50].

G. Searching for the symmetry basis

Structure of the mass matrix and its symmetries depend on the basis. In general, symmetry basis can differ substantially from the flavor basis considered so far. Therefore identification of the symmetry basis is crucial for uncovering the underlying physics.

Unfortunately, there is no clear guideline on how to search for this basis. One can perform a continuous change of the basis searching for situations when both neutrino and charge lepton mass matrices have certain common symmetries. Some hints can be obtained from explicit models constructed.

It is not excluded that symmetry basis coincides with the flavor basis and models of this type exist. One can expect that in this case symmetries based on some combinations of L_e, L_μ, L_τ play an important role.

The symmetry basis may not coincide but be close to the flavor basis. They can differ by rotation on the angle of the order of Cabibbo angle $\theta_C \sim \sqrt{m_\mu/m_\tau}$. Strong hierarchy of masses of charged leptons would suggest such a possibility.

At the same time the symmetry basis can strongly differ from the flavor basis. In some models, as a consequence of symmetry, the neutrino mass matrix is diagonal and maximal mixing comes from diagonalization of the charged lepton mass matrix. Further studies in this direction are necessary.

IV. NEUTRINOS AND NEW SYMMETRIES OF NATURE

Probably the most striking and unexpected outcome of the bottom up approach is indication of particular symmetries in the neutrino sector, that is, symmetries of the lepton mixing matrix and neutrino mass matrix in the flavor basis. Unusual thing is that symmetry is associated somehow with neutrinos and it does not show up in other sectors of theory. Several observations testify for such a "neutrino" symmetry(ies):

- Maximal or close to maximal 2-3 mixing;
- Zero or very small 1-3 mixing;
- Special value of 1-2 mixing;
- Hierarchy of mass squared differences;
- Quasidegenerate mass spectrum, if confirmed.

As far as last item is concerned, independent confirmation of the Heidelberg-Moscow positive result is needed. One should add that in physics often large or maximal mixing is related to degeneracy. So, possibility of the degenerate spectrum does not look implausible. ‘

The observations listed above could be hints of certain symmetries of the mixing and mass matrices. Some of these features can originate from the same underlying symmetry.

A. $\nu_\mu - \nu_\tau$ symmetry

Both maximal 2-3 mixing and zero 1-3 mixing indicate the same underlying symmetry and therefore deserve special attention. They are consequences of the $\nu_\mu - \nu_\tau$ permutation symmetry of the neutrino mass matrix in the flavor basis [39]. General form of such a matrix is

$$M = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix}. \tag{53}$$

This symmetry can be a part of discrete S_3 or D_4 groups and also can be embedded into certain continuous symmetries. Apparently matrices in eq.(36) for $c = 1$ and $a = b$, in eq.(40) for $x = y$ and $c = s$ are special cases of the general matrix eq.(53).

At first sight, one might consider this problematic since this symmetry can not be extended to the charged lepton sector. To see this, note that in the flavor basis for the charged leptons we have zero off diagonal elements $D_l = B_l = 0$, and therefore: $M_l = diag(A_l, C_l, C_l)$

in contradiction with $m_\mu \ll m_\tau$. If $D_l \neq 0$, one can get the required mass hierarchy. However in this case the charged lepton mass matrix also produces maximal mixing rotation. Neutrino and charged lepton rotations cancel leading to zero lepton mixing. Furthermore, for nonzero D_l the symmetry basis does not coincide with the flavor basis.

One way to resolve the problem is to have the Higgs bosons, which violate $(\mu - \tau)$ symmetry, couple weakly with neutrinos, but strongly - with the charge leptons. Examples where this is achieved using simple auxiliary symmetries like Z_2, have been discussed in the literature [60]. The difference of charged leptons and neutrinos appears because the right handed components l_R and ν_R have different transformation properties under Z_2. This is not unnatural in the context of supersymmetric theories where charged leptons get mass from the down Higgs whereas the neutrinos get mass from the up Higgs doublet. Such models can be embedded into supersymmetric SU(5) grand unified theories [61].

Other possibilities are to introduce the symmetry basis which differs from the flavor basis (in this case the symmetry will be 2-3 permutation symmetry), or to use other (approximate) symmetries which in the flavor basis are reduced to $\nu_\mu - \nu_\tau$ permutation.

Small breakings of these symmetries would manifest in the appearance of a small but nonzero θ_{13}. The question of course is what is small? A reasonable point of view to take is that if $\theta_{13} \sim r_\Delta$, then it is an indication of an underlying $\mu - \tau$ symmetry. However if $\theta_{13} \sim \sqrt{r_\Delta}$, no conclusion about this symmetry can be drawn since there are many examples where larger values of θ_{13} are possible.

Can this symmetry be extended to the quark sector? In [62] it is argued that in fact smallness of the 2-3 quark mixing, $V_{cb} \ll \sqrt{m_s/m_b}$, can also be a consequence of this symmetry.

There are two shortcomings of the discussed symmetry:

1). It does not determine masses: Symmetry fixes general form of the mass matrix (equalities of certain matrix elements) and not masses which are given by the values of the elements.

2). Symmetry does not determine the 1-2 sector.

So, one needs to use some more extended symmetries which involve all three generations. A widely studied such symmetry is the A_4 symmetry [44]. Other possibilities are: S_3[63], Z_4 [64], D_4 [65].

B. A symmetry example, A_4

An interesting class of models is based on the A_4 symmetry group of even permutations of 4 elements [44, 66, 67]. It is the symmetry of the tetrahedron and has the irreducible representations $\mathbf{3}$, $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$. The products of representations $\mathbf{3}\times\mathbf{3} = \mathbf{3}+\mathbf{3}+\mathbf{3}+\mathbf{1}+\mathbf{1}'+\mathbf{1}''$ and also $\mathbf{1}' \times \mathbf{1}'' \sim \mathbf{1}$ both contain invariant $\mathbf{1}$. This allows one to introduce the Yukawa couplings with special flavor structure. Furthermore, it is the existence of three different singlet representations which leads to substantial freedom to reproduce the observed pattern of masses and mixings. Notice also that A_4 is subgroup of $SO(3)$.

In all the models proposed so far three lepton doublets form the triplet of A_4: $L_i = (\nu_i, l_i) \sim \mathbf{3}$, $i = 1, 2, 3$. The right handed components of the charged leptons, l_i^c, neutrinos and Higgs doublets transform depending on model in different ways either as $\mathbf{3}$, or as $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$. Essentially the large (maximal) mixing originates from the fact that l_R and ν_R have different A_4 transformation properties.

A disadvantage of the model is that it also requires the introduction of new Higgs multiplets, and often new heavy leptons as well as quarks which are generic features of most symmetry approaches.

In A_4 models one should introduce Higgs fields with non-trivial A_4 transformation properties, that is in representations $\mathbf{3}$ and $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$. One can ascribe these properties to the SU_2 Higgs doublets, in which case 6 such doublets are required. Alternatively, one can keep SM Higgs doublet to be singlet of A_4 but introduce new SU(2) singlet fields which form non-trivial representations of A_4 (in the spirit of Froggatt-Nielsen approach). The latter however requires introduction of non-renormalizable operators (see e.g. [68]) or explicitly new heavy leptons and quarks [66].

There are different versions of the A_4 models. By appropriate choice of the Higgs fields and their VEV and/or right handed neutrino couplings one can obtain the tri-bimaximal mixing. The models constructed are based on the fact that tri-bimaximal mixing is given by the product of the trimaximal ("magic") rotation U_{tm} and maximal 1-3 rotation:

$$U_{tbm} = U_{tm} U_{13}^m. \tag{54}$$

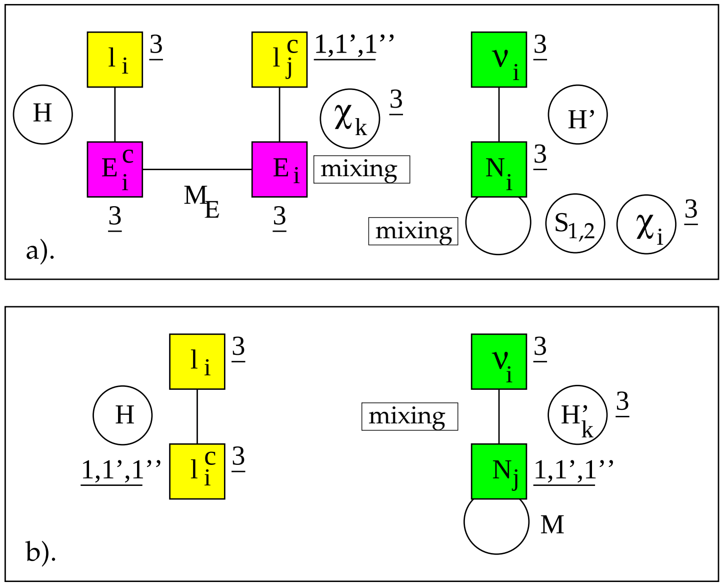

FIG. 6: Generation of the lepton masses and mixing in two models based on A_4: a). model [69] and b). model [70]. Lepton and Higgs multiplets are in shadowed boxes and circles. Numbers at the boxes and circles indicate the A_4 representations of the corresponding particles. The lines show the Yukawa couplings or bare mass terms in the models. We indicate also places where mixing is generated.

Here

$$U_{tm} \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad \omega \equiv e^{-2i\pi/3}. \tag{55}$$

As an illustration, in fig. 6 we show schemes of generation of the neutrino and charge lepton masses which lead to tri-bimaximal mixing in two models based on A_4.

In the model a) [69] which is certain modification of the early proposal [44, 66], the two Higgs doublets H and H' are invariant under A_4, and the SM singlets ξ form A_4 multiplets. New heavy leptons E, E^c have to be introduced. The model content and transformation properties of the fields have been arranged in such a way that in the symmetry basis the charged lepton mass matrix produced the U_{tm} rotation, whereas the neutrino mass matrix produces maximal 1-3 mixing [67]. The letter requires also certain VEV alignment.

In the model b) [70] the SU_2 Higgs doublets form the triplet and singlets of A_4.

This illustrates generic problems and complexity of realization of "neutrino symmetries".

C. Real or accidental

The main question here is whether the symmetries hinted by the observations are simply accidental or they have real physics behind them specially in view of the fact that the price one pays in terms of number of Higgs fields and/or extra leptons for constructing a realistic model can be very heavy. So, a fruitful approach is to look for possible deviations from symmetries in data (*e.g.* small θ_{13} along with correlation between θ_{13} and $\theta_{23} - \pi/4$) and explore models that may give other tests, because if symmetries are not accidental, they have consequences of fundamental importance. New structures are predicted, unification path may differ substantially from what we are using now, etc. etc.. The symmetries may hold the key to understanding of the difficult flavor problem.

Another possibility is that the symmetries are not accidental but the underlying theory has not been found yet and observed symmetry relations are hints of a new sector in physics, *e.g.* flavor universal mixing with new singlets may produce symmetric contribution to the active neutrino mass matrix (sec. 8).

The only way to establish that symmetry is not accidental is to find new consequences of the symmetry -i.e. to make predictions in the context of a certain model and to test the predictions in experiment. It is important to find not just one but several predictions (see discussion in sec. 5.5).

Finally let us stress that the observational basis for the existence of symmetries (real or accidental) is not yet well established. As we described in sec. 2.4, still significant deviation from maximal 2-3 mixing is possible and 1-3 mixing could be relatively large. So, further experimental measurements will be decisive.

V. LEPTONS AND QUARKS

Joint consideration of quarks and leptons and searches for possible relations between quark and lepton parameters are of fundamental importance, since (i) this may provide a unified clue for understanding fermion masses and mixings and (ii) give more insight into the unification of particles and forces in Nature.

In what follows we will compare quark and lepton masses and mixing and will discuss various ideas about possible relations of quarks and leptons such as (i) quark-lepton symmetry;

(ii) quark-lepton unification; (iii) quark-lepton universality; (iv) quark-lepton complementarity.

A. Comparing leptons and quarks

Apparently there are strong differences between the masses and mixing in the quark and lepton sectors (fig. 7). The ratios of masses of neutrinos and the corresponding upper quarks are $m_2/m_c < 10^{-10}$, $m_3/m_t < 10^{-12}$. Lepton mixings are large, quark mixings are small. The 1-2 mixing is the largest for quarks, whereas 2-3 is the largest lepton mixing. The only common feature is that the 1-3 mixing (mixing between the remote generations) is small in both cases.

More careful consideration however reveals some interesting features: It seems the 1-2 as well as 2-3 mixing angles in the quark and lepton sectors are complementary in the sense that they sum up to maximal mixing angle [71, 72, 73]:

$$\theta_{12} + \theta_C \approx \frac{\pi}{4}, \tag{56}$$

$$\theta_{23} + V_{cb} \approx \frac{\pi}{4}. \tag{57}$$

While for various reasons it is difficult to expect exact equality in the above relations, qualitatively one can say that there is a certain correlation: 2-3 mixing in the lepton sector is close to maximal because the corresponding quark mixing is small, the 1-2 mixing deviates from maximal substantially because 1-2 (Cabibbo) quark mixing is relatively large. For the 1-3 angles we do not see simple connection, and apparently the quark relation $\theta_{13} \sim \theta_{12} \times \theta_{23}$ does not work in lepton sector. Below we will explore possible meaning of the above relations called quark-lepton complementarity (QLC) in literature [71].

Comparing the ratio of neutrino masses eq.(23) with ratios for charged leptons and quarks (at m_Z scale):

$$\frac{m_\mu}{m_\tau} = 0.06, \quad \frac{m_s}{m_b} = 0.02 - 0.03, \quad \frac{m_c}{m_t} = 0.005 \tag{58}$$

one concludes that neutrino hierarchy (if exists at all) is the weakest one. This is consistent with possible mass-mixing correlation, so that large mixings are associated a weak hierarchy: $\sqrt{m_i/m_j} \sim \theta_{ij}$.

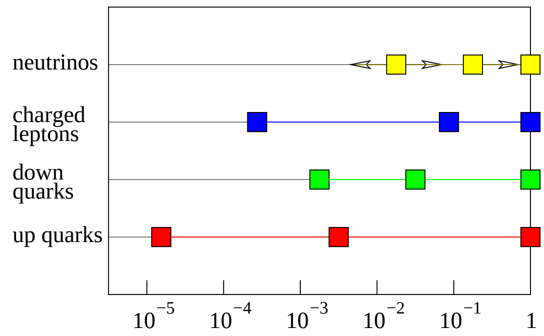

FIG. 7: Mass hierarchies of quarks and leptons. The mass of heaviest fermion of a given type is taken to be 1.

It is also intriguing that

$$\sqrt{\frac{m_\mu}{m_\tau}} \approx \sin\theta_{12}^q \approx \sin\theta_C. \tag{59}$$

Does this perhaps indicate that the Cabibbo angle is the universal flavor parameter for both quark and lepton physics? In a class of grand unified models to be discussed later, Cabibbo angle, indeed, becomes the key parameter of the neutrino mass matrix and describes the neutrino masses as well as mixings.

In fig. 7 we show the mass ratios for three generations. The strongest hierarchy and geometric relation $m_u \times m_t \sim m_c^2$ exist for the upper quarks. Apart from that no simple relations show up. It looks like the observed pattern is an interplay of some regularities - flavor alignment and randomness - "anarchy". Below we explore possible meaning behind this picture.

B. Quark-lepton symmetry

There are good reasons to suspect that quarks and leptons may be two different manifestations of the same form of matter. The first hint for that arises from the observed similarities between weak interaction properties of quarks and leptons. Each quark has its own counterpart in the leptonic sector which has the same weak isospin properties: u_L corresponds to ν_L, d_l - e_L, *etc.*. This is generally known as quark-lepton symmetry and even though it manifests itself only in the left-handed helicity sector of quarks and leptons, it is often considered as a hint of further unification among these two very different kinds of matter. It can be extended to the RH sector, when leptons are treated as the 4th color [74] following Pati-Salam $SU(4)_C$ unification symmetry.

The second hint comes from the attractive hypothesis of grand unification of matter and forces which argues that at very short distances, all forces and all matter unify. In Grand Unified theories (GUT's) quarks and leptons form multiplets of the extended gauge group. The most appealing such group is SO(10) where all known components of quarks and leptons including the RH neutrinos fit into the unique 16-plet spinor multiplet [75]. It is difficult to believe that all these features are accidental.

However different patterns of masses and mixing at first sight strongly break the quark-lepton symmetry. Furthermore, if particular symmetries are found to exist only in the leptonic sector, they may indicate that quarks and leptons are fundamentally different.

C. Quark-lepton universality

In spite of strong difference of masses and mixings of quarks and leptons we still can speak about the approximate quark-lepton symmetry or even universality. The universality is realized in terms of mass matrices or matrices of the Yukawa couplings and not in terms of observables - mass ratios and mixing angles. The point is that very similar mass matrices can lead to substantially different mixing angles and masses (eigenvalues) if the matrices are nearly singular (approximately equal matrices of rank-1) [76]. The singular matrices are "unstable" in the sense that small perturbations can lead to strong variations of mass ratios and (in the context of seesaw) mixing angles. The well known examples of singular matrices are the "democratic" mass matrix [77] and the matrix with only one non-zero element m_{33}.

Let us consider the universal structure for the mass matrices of all quarks and leptons

$$Y_u \approx Y_d \approx Y_D \approx Y_M \sim Y_L \approx Y_0. \tag{60}$$

We can assume that in the zeroth order of some approximation all fermion mass matrices are equal to the same universal singular matrix Y_0. In eq.(60) Y_D is the Dirac type neutrino Yukawa matrix. The Majorana type matrix for the RH neutrinos, Y_M, can in general differ from the others being *e.g.* as $Y_M \approx Y_0^2$.

As an important example we can take

$$Y_0 = \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda \\ \lambda^2 & \lambda & 1 \end{pmatrix}, \qquad \lambda \sim \sin\theta_C \sim 0.2 - 0.3. \tag{61}$$

This matrix has only one non-zero eigenvalue and no physical mixing appears since matrices for all fermions are diagonalized by the same rotation. In this respect all singular matrices are equivalent (up to basis definition) till corrections are introduced - corrections break the equivalence.

Let us introduce perturbations of matrix structure, ϵ, in the following form

$$Y^f_{ij} = Y^0_{ij}(1 + \epsilon^f_{ij}), \quad f = u, d, e, \nu, N, \tag{62}$$

where Y^0_{ij} is the element of the original singular matrix. This form can be justified, *e.g.*, in the context of the Froggatt-Nielsen mechanism [78]. It turns out that small perturbations $\epsilon \leq 0.25$ are enough to reproduce all large differences in mass hierarchies and mixings of quarks and leptons [76].

Smallness of neutrino masses is explained by the seesaw mechanism. Nearly singular matrix of the RH neutrinos which appears in the denominator of the seesaw formula leads to enhancement of lepton mixing and can flip of the sign of mixing angle which comes from diagonalization of the neutrino mass matrix. So the angles from the charged leptons, U_ℓ, and neutrinos, U_ν, sum up, whereas in quark sector mixing angles from up and down quark mass matrices subtract leading to small quark mixing.

Notice that different mass hierarchies of the up and down quarks (as well as charged leptons) may testify that two different universal matrices should be introduced for fermions with the third projection of weak isospin 1/2 and -1/2. This can also be related to existence of two different Higgs doublets giving masses to those components (as in supersymmetric theories).

Instead of mass matrices one can consider the universality of rotation (mixing) matrices V_U which diagonalize mass matrices of all fermions in certain basis. The model can be arranged in such a way that in the lowest order $U_{CKM} = V_U^\dagger V_U = I$, whereas $U_{PMNS} = V_U^T V_U$ contains large mixings [79].

In connection to the q-l universality one can consider the following working hypothesis:

1). No particular "neutrino" symmetry exists, and in general one expects some deviation of the 2-3 mixing from maximal as well as non-zero 1-3 mixing. Nearly maximal 2 -3 mixing would be accidental in this case. Instead, some family symmetry is realized which ensure universality of mass matrices and their particular structure.

2). Seesaw mechanism with the scale of RH neutrino masses $M \sim (10^7 - 10^{15})$ GeV explains smallness of neutrino mass.

3). The quark-lepton unification or Grand Unification are realized in some form, *e.g.*, $SO(10)$.

4). The quark-lepton symmetry is (weakly) broken with some observable consequences like $m_b \approx m_\tau$.

5). Large lepton mixing is a consequence of the seesaw mechanism - seesaw enhancement of lepton mixing (special structure of the RH neutrino mass matrix), or/and due to contribution from the type II seesaw (which we will consider in sect. 7.1). Flavor symmetry or/and physics of extra dimensions could determine this special structure.

D. Quark-lepton complementarity

As noted in eqs. (56) and (57), the latest determination of solar mixing angle gives for the sum

$$\theta_{12} + \theta_C = 46.7^\circ \pm 2.4^\circ \quad (1\sigma) \tag{63}$$

which is consistent with maximal mixing angle within 1σ (fig. 1). The fact that for the 2-3 mixings the approximate complementarity is also fulfilled hints some more serious reasons than just numerical coincidence [179]. A possibility that the lepton mixing responcible for solar neutrino conversion equals maximal mixing minus θ_C was first proposed in [81], and corrections of the bimaximal mixing by the CKM type rotations discussed in [82].

If not accidental, the quark-lepton complementarity would require certain modification of the picture described in the previous section [72, 73, 83]. It implies the existence of some additional structure in the leptonic (or quark?) sector which generates bi-maximal mixing. In this sense it might indicate a fundamental difference between leptons and quarks. At the same time there should be the quark- lepton unification which communicates the quark mixing to the lepton sector. A general scheme could be that [72, 73]

$$\text{“lepton mixing = bimaximal mixing} - \text{CKM}''. \tag{64}$$

(Another option: "CKM = bimaximal - PMNS" may have different implications).

There is a number of non-trivial conditions for the *exact* QLC relation to be realized [73].

1). Order of rotations: apparently the matrices U_{12}^{m} and $U_{12}^{CKM\dagger}$ should be multiplied in the following order:

$$U_{PMNS} \equiv U_L^{\dagger} U_{\nu} = ...U_{23}^{m}...U_{12}^{m} U_{12}^{CKM\dagger} \tag{65}$$

(last two matrices can be permuted). Different order leads to corrections to the exact QLC relation.

2). Matrix with CP violating phases should not appear between $U_{12}^{CKM\dagger}$ and U_{12}^{m} or the CP violating phase in this matrix should be small enough [73, 84].

3). The renormalization group effects should be small since presumably the quark-lepton symmetry which leads to the QLC relation is realized at high mass scales.

Let us first describe two possible (to some extend extreme) scenarios of eq.(64) which differ by origin of the bi-maximal mixing and lead to different predictions.

(1). QLC-1: In the symmetry basis maximal mixing is generated by the neutrino mass matrix: $U_{\nu} = U_{bm}$; it can be produced by the seesaw mechanism. The charged lepton mass matrix gives the CKM rotation $U_{\ell} = U_{CKM}^{\dagger}$, as a consequence of the q-l symmetry: $m_l = m_d$. In this case the order of matrices eq.(65) is not realized (U_{12}^{CKM} should be permuted with U_{23}^{m}) and consequently the QLC relation is modified: $\sin\theta_{12} = (1/\sqrt{2})\cos\theta_C - 0.5\sin\theta_C$ or

$$\sin\theta_{12} \approx \sin(\pi/4 - \theta_C) + 0.5\sin\theta_C(\sqrt{2} - 1). \tag{66}$$

Numerically we find $\sin^2\theta_{12} = 0.331$ which is practically indistinguishable from the tri-bimaximal mixing prediction. The predicted 1-3 mixing, $\sin\theta_{13} = \sin\theta_C/\sqrt{2}$, is close to the upper experimental bound (fig. 3). Combining this with expression for 1-2 mixing we get an interesting relation $\theta_{12} \approx \pi/4 - \theta_{13}$ [84].

2). QLC-2: In the symmetry basis maximal mixing comes from the charged lepton mass matrix, $U_{\ell} = U_{bm}$, and the CKM, $U_{\nu} = U_{CKM}^{\dagger}$, appears from the neutrino mass matrix due to the q-l symmetry: $m_D \sim m_u$ (assuming also that in the context of seesaw the RH neutrino mass matrix does not influence mixing, *e.g.*, due to "factorization"). In this case the QLC relation is satisfied precisely: $\sin\theta_{12} = \sin(\pi/4 - \theta_C)$. The 1-3 mixing is very small - of the order V_{ub}.

According to fig. 1 the best fit experimental value of θ_{12} is in between the QLC-1 and QLC-2 predictions and further measurements of the angle with accuracy $\Delta\theta_{12} \sim 1^\circ$ are required to disentangle the scenarios.

Other possibilities exist too, for instance one maximal mixing may come neutrino mass matrix in the symmetry basis and another one from charge lepton mass matrix.

There are two main issues related to the QLC relation:

- origin of the bi-maximal mixing;

- mechanism of propagation of the CKM mixing from the quark to lepton sector.

The main challenge here is that the required quark-lepton symmetry is broken. In particular, the leptonic mass ratio $m_e/m_\mu = 0.0047$ is much smaller than the quark ratio $m_d/m_s = 0.04 - 0.06$; also masses of muon and s-quark are strongly different at the GUT scale.

Precise QLC relation may imply that

- the q-l symmetry is actually weakly broken as we discussed in sec. 5.3;

- the q-l symmetry is very weakly broken for up quarks and neutrinos in a sense that for Dirac matrices $M_u \approx M_D$. Then CKM propagates via the up-sector;

- the breaking affects mainly the masses and mass ratios but not mixings.

Anyway, the mass matrices are different for quarks and leptons and "propagation" of the CKM mixing leads to corrections to the QLC relation at least of the order $\Delta\theta_{12} \sim \theta_C m_d/m_s \sim 0.5 - 1.0^\circ$ [73].

Consider the case of QLC-1 (bimaximal mixing from neutrinos), where deviation of quark mixing from zero and lepton mixing from maximal follow from the down quarks and leptons. If the leptonic mass matrices has similar structure to the d-quark mass matrix with Gatto-Sartori-Tonin (GST) relation one would expect $\theta_l \sim \sqrt{m_e/m_\mu} \approx \theta_C/3$ and deviation from maximal mixing $\theta_l/\sqrt{2} = 1/3\sqrt{2}\theta_C$ turns out to be too small [85]. There are several proposals to enhance the shift angle. In particular, the neutrino mass matrix can be modified $\mathcal{M}_\nu = \mathcal{M}_{bm} + \delta\mathcal{M}$, where $\mathcal{M}_{bm}$ produces the bi-maximal mixing and δm leads to deviation [85]. In particular, δm can be due to the seesaw type-II contribution [86]. However in this case connection to quark mixing is lost and the relation eq.(56) is simply accidental. Notice that the ratio of the mass squared differences, $r_\Delta \sim \sin\theta_C$, so that the shift, θ_l, can be related simply with generation of the solar mass split and therefore be of purely leptonic origin.

In the context of quark-lepton symmetric models, the enhancement may have the group theoretical origin. In [87] for certain operators generating fermion masses the relation $\theta_l = 3\theta_C/2$ has been found, where factor 3/2 is the ratio of Clebsh-Gordon coefficients.

The renormalization group effects on 1-2 mixing are in general small and furthermore they lead to increase of the angle θ_{12} at low scales. The negative shift can be obtained from renormalization group effects in presence of the non-zero 1-3 mixing [73]. Also the threshold corrections due to some intermediate scale physics like low scale supersymmetry (MSSM) can produce the negative shift thus enhancing the deviation from maximal mixing [88]. Strong shift can also be obtained from RGE effects between and above the seesaw scales related to the RH neutrino masses (see sec. 6.3) [50].

To avoid the additional 1/3 suppression of θ_l one can abandon the GST-type relation for charged leptons. Then $\theta_l \sim \theta_C$ would imply nearly singular character of the 1-2 leptonic submatrix.

As remarked before, quark-lepton symmetry can propagate θ_C to lepton sector exactly if the neutrino mass matrix is the source of both bi-maximal mixing and the CKM rotations. The charged lepton and down quark mass matrices should be diagonal, and as a consequence of the q-l symmetry, $m_u = m_D$. The left rotations for these matrices give U_{CKM} and the rest of the seesaw structure generates the bimaximal mixing. In this case, however, the GST-relation in the quark sector becomes accidental. If the bi-maximal mixing is generated by charge leptons (lopsided scenario, see sec. 6.4.4) the QLC relation becomes precise [72].

The role of CP-phases can be important in the q-l relations [73]. CP violating phase in U_{CKM} produces very small effect on QLC due to smallness of V_{ub}. Also in this scenario the leptonic CP phase is very small. On the other hand appearance of the phase matrices in between U_{12}^{CKM} and U_{12}^{m} will both modify the QLC and the leptonic Dirac phase, δ. Apparently, the relation between these two modifications should appear. In the QLC-1 scenario insertion of the phase matrix $I_{phase} \equiv diag(e^{i\alpha}, e^{i\beta}, 1)$, between two 1-2 rotations: $U^{CKM\dagger} I_{phase} U_{bm}$ leads to the following change of the QLC relation [84]:

$$\theta_{12} \approx \frac{\pi}{4} - \frac{\theta_C}{\sqrt{2}} \cos(\delta - \pi). \tag{67}$$

So, the phase diminishes the shift, thus destroying the relation. Maximal shift required by QLC implies $\delta \approx \pi$, that is, suppressed CP violation phase.

There are few attempts to construct consistent quark-lepton model which reproduces the QLC relation.

The simplest possibility is the $SU(2)_L \times SU(2)_R \times SU(4)_C$ model which implements the quark-lepton symmetry in the most straightforward way [83, 87]. The strategy is to obtain (using an additional flavor symmetry) the neutrino mass matrix with inverted mass hierarchy which leads naturally to the bimaximal mixing (QLC-1 realization). The quark-lepton symmetry provides equality of the mass matrices $M_l = M_d$, and consequently the same CKM type mixing in both sectors. The perturbations to the matrices, δM_l and δM_d, should be introduced which break the q-l symmetry and correct the masses. (In [83] they are due to the non-renormalizable operators with new Higgs fields which transform as **15** of $SU(4)_C$). These corrections however modify relation θ_l and θ_C, and equality is matter of tuning of continuous parameters. Another possibility [87] is to introduce the non-renormalizable operators which include couplings with Higgs in **4** of $SU(4)_c$ as well as singlet flavon fields *a la* Froggatt-Nielsen. Selecting particular type of operators one can get inequality of matrices M_l and M_d already in the lowest order and enhance the leptonic angle: *e.g.* like $\theta_l = 3/2\theta_C$, as we have marked previously. The enhancement allows to reproduce the QLC relation eq.(56) almost precisely.

Different approach to resolve the problem of decoupling of masses and mixing is to use non-abelian flavor symmetries [72]. Via minimization of the potential the symmetries lead to zero or to maximal (bi-maximal) mixing independently of the mass eigenvalues.

The Cabibbo mixing can be transmitted to the lepton sector in more complicated way (than via the q-l symmetry). In fact, $\sin\theta_C$ may turn out to be a generic parameter of theory of fermion masses - the “quantum” of flavor physics, and therefore to appear in various places: mass ratios, mixing angles. The relation eq.(59) is in favor of this possibility.

On the other hand, the same relation eq.(59) may suggest that the QLC relation is accidental. Indeed, it can be written and interpreted as pure leptonic relation

$$\theta_{12} + \theta_{\mu\tau} = \frac{\pi}{4}, \quad \tan\theta_{\mu\tau} \equiv \sqrt{\frac{m_\mu}{m_\tau}}. \tag{68}$$

This relation may even be more difficult to realize in models.

Following an idea that $\lambda \approx \sin\theta_C$ is the “quantum” of the flavor physics one can consider the Cabibbo angle as an expansion parameter for mixing matrices. In zero approximation the quarks have unit mixing matrix: $U^0_{CKM} = I$, whereas leptons have $U^0_{PMNS} = U_{bm}$

or bi-large mixing matrix. The λ size corrections can be included as $U_{PMNS} = U_\lambda^\dagger U_{bm}$ or $U_{PMNS} = U_{bm} U_\lambda^\dagger$. Interesting possibility (in a spirit of the QLC relation) is that $U_\lambda = U_{CKM}(\lambda)$ in the Wolfenstein parameterization [82, 89]. In this case one gets universal description (parameterization) of quark and lepton mixing matrices. This apparently reduces the number of free parameters in the problem and also establishes various relations between mixing angles.

In general on can take U_λ as a matrix with all three λ size rotations and study properties of the PMNS matrix obtained by insertion of the U_λ in various places of the zero order structure [90]. That is, $U_\lambda^\dagger U_{bm}$, $U_{bm} U_\lambda^\dagger$ or $U_{23}^m U_\lambda^\dagger U_{12}^m$, *etc.*.

E. Empirical relations

As we have mentioned before, establishing the empirical relations between masses and mixings of fermions may give a clue to the underlying physics. The tri-bimaximal mixing scheme and QLC equality are examples of relations "between mixings without masses". One should note, of course, the GST relation $\sin\theta_C \approx V_{us} \approx \sqrt{m_d/m_s}$ [91], and $m_d/m_s = \sqrt{m_u/m_c}$ which determine substantially the form of quark mass matrices, *etc.*

A particularly intriguing such relation is the Koide relation [92, 93, 94] according to which the pole masses of charge leptons satisfy the equality

$$Q_l \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \tag{69}$$

is satisfied with accuracy 10^{-5}:

$$Q_l^{(pole)} = 2/3 \ {}^{+0.00002}_{-0.00001}. \tag{70}$$

The Koide formula eq.(69) is interesting not only because of precision but also because it was obtained in the context of certain model - the composite model of the leptons [93] and not as empirical relation. In fact, it allowed to predict precise value of the tau-lepton mass.

There are several properties of the relation eq.(69) which could have interesting implications [95].

(i) Varying masses one finds that the minimal value, $Q_{min} = 1/3$, corresponds to the degenerate spectrum and the maximal one, $Q_{max} = 1$ - to the strongly hierarchical spectrum. So, the quantity Q_l is a good measure of degeneracy of spectrum. The experimental value $2/3$ is exactly in between the two extremes.

(ii) The relation involves 3 generations explicitly. The mass of electron can not be neglected and therefore in the underlying theory m_e can not be considered as perturbation. In fact, the value 2/3 may be interpreted as $2/N_f$, where $N_f = 3$ is the number of flavors.

(iii) The formula is invariant under interchange of flavors $e \leftrightarrow \mu$, *etc.*, and therefore implies S_3 (or wider) underlying symmetry. The value 2/3 may have certain group theoretical origin.

(iv) Essentially eq.(69) gives relation between the two mass hierarchies $r_e \equiv m_e/m_\tau$, $r_\mu \equiv m_\mu/m_\tau$, and does not depend on the absolute scale of masses:

$$Q_l = \frac{1 + r_e + r_\mu}{(1 + \sqrt{r_e} + \sqrt{r_\mu})^2} = \frac{2}{3}. \qquad (71)$$

So, it can be realised for different sets of hierarchies.

(v) The formula may have certain geometrical origin [96, 97]. Introducing vectors $\vec{M} = (1, 1, 1)$ and $\vec{L} = (\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ we can rewrite it as

$$Q_l = \frac{1}{3\cos^2\theta_{ML}}, \qquad \cos\theta_{ML} \equiv \frac{\vec{L} \cdot \vec{M}}{|\vec{L}||\vec{M}|}. \qquad (72)$$

Apparently the experimental result corresponds to $\theta_{ML} = 45°$.

(vi) The relation has bilinear structure in $\sqrt{m}$ which may imply that masses are bilinear of some other physical quantities: coupling constants or VEV's. In fact, the Koide relation is reproduced if

$$m_i = m_0(z_i + z_0)^2, \qquad (73)$$

where $z_0 = \sqrt{\sum_i z_i^2/3}$ and $\sum_i z_i = 0$. Such a situation can be realized in the case of the radiative mechanism of mass generation: in one loop $m \propto Y^2$, or in the seesaw mechanism $m \propto \mu\mu' M^{-1}$.

(vii) The quantity Q_l is not invariant under RGE running. At the Z^0 - mass $Q_l(m_Z) \approx 1.002 Q_l^{(pole)}$ [98, 99]. Above m_Z the renormalization is negligible in the SM, and it can lead to further increase of Q_l by about 0.7% in MSSM at $M_R \sim 10^{14}$ GeV and for $\tan\beta = 50$ [99]. So, the renormalization effect is much larger than the error bars in eq.(70) and therefore Q_l deviates from 2/3 at high scales (already at the EW scale). This may indicate various things: the relation is accidental; the accuracy for the pole masses is accidental; physics responsible for the relation, and therefore the lepton masses, is at low scales.

(viii) The relation eq.(69) is not universal: it can still be valid for the down quarks: $Q_d \sim 0.7$ at m_Z, but it is certainly violated for the up quarks: $Q_u \sim 0.9$.

(ix) Important aspect is that the mass relation does not depend on mixing. That is, physics of mass generation and that of mixing should decouple. Mixing can be included if the relation $Q = 2/3$ is considered, *e.g.*, for the "pseudo-masses" introduced as $\tilde{m}_\alpha \equiv \sum_i U_{L\alpha i} m_i$, where U_L is the matrix of rotation of the LH components which diagonalizes the mass matrix in the "symmetry" basis [100]. For charge leptons one should take $U_L^l = I$. For quarks one can select the matrices so that universality $Q^u = Q^d = 2/3$ is restored.

Till now no realistic and consistent model for the Koide relation is constructed (see [95] for review). Among interesting proposals one should mention the radiative (one loop) mechanisms of charged leptons masses generation [92, 93]; the seesaw mechanism [94, 101]; mechanism based on the democratic mass matrices and S_3 symmetry [102]. An interesting possibility is that lepton masses are generated by bi-linear of VEV's of new scalar fields: $m_i \propto \langle\bar{\phi}_i\rangle\langle\phi_i\rangle$. Then as a consequence of symmetry of the scalar potential (S_3 and $SU(3)$ symmetries have been considered), the VEV's have the property $\langle\bar{\phi}_i\rangle \sim z_i + z_0$ in eq.(73) [95].

What about neutrinos? Due to weaker mass hierarchy eq.(22) neutrino masses do not satisfy the Koide relation. Depending on the unknown absolute mass scale one finds $Q_\nu = 0.33 - 0.60$ [97, 98, 99], where the lower bound corresponds to the degenerate spectrum and the upper one to $m_1 = 0$. The universality can be restored if one uses the pseudo-masses [100]. Notice that since $U_L^l = I$ for charged leptons, for neutrinos we have $U_L^\nu = U_{PMNS}$. Then from the condition $Q_\nu = 2/3$ one finds for the allowed region of neutrino oscillation parameters: $m_1 \sim (3 \pm 1)10^{-2}$ eV, $\theta_{12} > 35^\circ$ and $\theta_{23} > 50^\circ$. All neutrino masses are of the same order, and large lepton mixing is related to the absence of mass hierarchy *a la* the GST relation.

Another proposal is to modify the Koide relation for the upper quarks and neutrinos without introduction of mixing [98]. Observing that $Q_\nu < 2/3$ but $Q_u > 2/3$ one can assume a kind of mass complementarity $Q_l + Q_d = Q_\nu + Q_u$. That would lead to the lightest neutrino mass $m_1 \approx 10^{-5}$ eV.

Notice that in these considerations smallness of neutrino mass and its possible Majorana nature have not been taken into account. Apparently, the presence of the Majorana mass matrix of the RH neutrinos in the context of seesaw mechanism can influence the implications of the Koide relations for neutrinos. Alternatively one can imagine that mechanism responcible for smallness of the neutrino masses does not influence ratios of masses.

The question: "real or accidental" is still open; and the lesson is that just one very precise prediction confirmed by very precise measurements may not be enough to verify theory.

VI. SEESAW: THEORY AND APPLICATIONS

As already noted earlier, seesaw mechanism is one of the simplest ways to understand the small neutrino masses. It has important implications and connections to a number of fundamental issues which we will discuss in this section.

- What is the scale of M_R and what determines it?
- Is there a natural reason for the existence of the right handed neutrinos - the main element of seesaw?
- Is the seesaw mechanism by itself enough to explain all aspects of neutrino masses and mixings?
- On a phenomenological level, what is the flavor structure of the right handed neutrino sector? Can we determine it from purely low energy neutrino observations, for example?

A. RH neutrino masses and scale of seesaw

The scale of the seesaw (type I) is related to the scale of RH neutrino masses. Some idea about M_R can be obtained from the naive estimation of masses for the third generation:

$$M_R \sim k(M_R)\frac{m_D^2}{m_3} = k(M_R)\frac{m_t^2}{\sqrt{\Delta m_{23}^2}} \approx 5 \cdot 10^{14} \text{ GeV}, \tag{74}$$

where m_t is the top quark mass, $k(M_R)$ is the renormalization group factor of the D=5 operator. (Here we assume normal mass hierarchy.) It is this large scale which indicates that neutrino mass is related to new physics beyond that implied by the charged fermion masses. The scale eq.(74) is rather close to the GUT scale and in fact can be immediately related to the GUT scale. In this sense the smallness of the neutrino mass is the direct indication of GUT.

Situation is more complicated if one considers all three neutrinos and takes into account mixing among them. There is a number of uncertainties and ambiguities in determination of

M_R: (i) we do not know yet scale of light neutrino masses (which can change the estimation by about 1 order of magnitude); (ii) the Dirac masses of neutrinos are not known and one needs to make some assumptions; (iii) mixing can strongly influence the masses of RH neutrinos; (iv) it is not clear yet that seesaw type I gives main contribution to the neutrino mass. If it is subdominant, the masses of the RH neutrinos can be larger; (v) more than 3 singlet fermions (RH neutrinos) can be involved in generation of the light neutrino masses. In this case (as we will see later) the scale eq.(74) may turn out to be a "phantom" scale which does not correspond to any physical reality.

An assumption that $m_D \propto m_q$ typically leads to rather strong hierarchy of the RH neutrino masses: $M_{Ri} \propto m_{qi}$ or even stronger. Their values can spread in the interval $(10^5 - 10^{16})$ GeV, though in some particular cases two masses can be quasi degenerate.

To get small masses of usual active neutrinos it is enough to have only two RH neutrinos which means that the third one can be arbitrarily heavy: *e.g.*, at the GUT or even Planck mass scale.

B. Seesaw, B - L and L-R symmetries

What is physics content of this new scale? The seesaw scale can be identified as the scale of violation of certain symmetries. The fact that M_R is much smaller than the Planck scale is an indication in favor of this. It is therefore appropriate at this point to discuss possible origin of the RH neutrino masses.

To answer this question, it is important to note the changes that occur in the standard model with the addition of one right handed neutrino per generation: the most obvious change is that it restores the quark-lepton symmetry. But on a more fundamental level, it turns out that in the presence of three N's, the symmetry $B - L$ which was a global symmetry in the standard model becomes a gaugeable symmetry since one has the condition $Tr(B - L)^3 = 0$, which implies that gauge anomalies cancel. The gauge group of weak interactions expands to become the left-right symmetric group $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ [103] which is a subgroup of the $SU(2)_L \times SU(2)_R \times SU(4)_c$ group introduced by Pati and Salam [74]. This leads to a picture of weak interactions which is fundamentally different from that envisaged in the standard model in that *weak interactions like the strong and gravitational interactions becomes parity conserving.* Furthermore, in this theory, the electric charge formula is given by [104]:

$$Q \; = \; I_{3L} \; + \; I_{3R} \; + \; \frac{B-L}{2}, \tag{75}$$

where each term has a physical meaning unlike the case of the standard model. When only the gauge symmetry $SU(2)_R \times U(1)_{B-L}$ is broken down, one finds the relation $\Delta I_{3R} \; = -\Delta\left(\frac{B-L}{2}\right)$. This connects $B-L$ breaking, *i.e.* $\Delta(B-L) \neq 0$, to the breakdown of parity symmetry, $\Delta I_{3R} \neq 0$. It also reveals the true meaning of the standard model hypercharge as $\frac{Y}{2} \; = \; I_{3,R} \; + \frac{B-L}{2}$.

To discuss the implications of these observations for seesaw mechanism, note that in stage I, the gauge symmetry is broken by the Higgs multiplets $\Delta_L(3,1,2) \oplus \Delta_R(1,3,2)$ to the standard model and in stage II by the bidoublet $\phi(2,2,0)$. In the first stage, the right handed neutrino picks up a mass of order $f < \Delta_R^0 > \equiv f v_R$, then ϕ produces the Dirac mass term. The presence of coupling of the triplets with bidoublet $\lambda \Delta_L \Delta_R^\dagger \phi\phi$ leads to the shift of minimum of potential from $\Delta_L = 0$, so that this triplet acquires the so called "induced VEV" from the diagram in fig.8b, of the size

$$v_L = \langle \Delta_L^0 \rangle = \frac{\lambda v_{wk}^2}{v_R}. \tag{76}$$

As a consequence the mass matrix (8) is generated with the components

$$m_L = f v_L, \quad m_D = h v_{wk}, \quad M_R = f v_R. \tag{77}$$

The light neutrino mass matrix can be then written as

$$\mathcal{M}_\nu = \frac{v_{wk}^2}{v_R}(\lambda f - h^T f^{-1} h). \tag{78}$$

Important point to note is that v_L and the see-saw type II term are suppressed by the same factor as the seesaw type I contribution, so that the overall seesaw suppression remains [6]. As a consequence of the L-R symmetry, the two contributions are partly correlated: both depend on the same matrix f.

Important problem is to reconstruct f from the low energy data. In this connection, an interesting property of the formula eq.(78) in the lowest order approximation (before RGE corrections) is the "seesaw duality"[105]: for any solution f, a dual solution $\tilde{f} = m_\nu / v_L - f$ exists. In the limit of very large right handed neutrino masses, the general seesaw formula for neutrino masses reduces to the triplet seesaw (type II) formula.

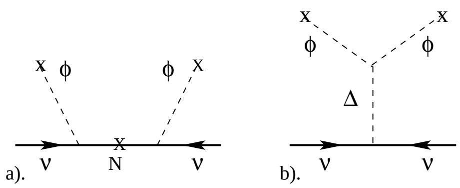

FIG. 8: Feynman diagrams for a). type I and b). type II seesaw mechanisms.

Note that the scale of the right handed neutrino masses (or the seesaw scale) is now the scale of B-L breaking, which shows that it is not the Planck scale as would be the case if the left-right symmetry group was not considered. We will see in a later section that in the context of SO(10) grand unified theories which embed the left-right model, the seesaw scale can indeed be the GUT scale, removing one arbitrariness in the description of neutrino masses.

C. Seesaw and the RGE effects

As we saw in sec. 6.1 the masses of RH neutrinos are in general substantially smaller than the GUT scale. Furthermore, typically they have an extremely large spread (often related to the quadratic mass hierarchy of quarks): from 10^6 to 10^{15} GeV. That determines important features of the RGE effects if the flavor physics (structure of the Yukawa couplings) is fixed at $M_f \geq M_{GUT}$. In this case there are three different energy regions with different RGE behavior:

1). Region below the seesaw scales, $\mu < M_1$ where M_1 is the mass of the lightest RH neutrino. The RGE effects in this region have been studied in sec. 3.6.

2). Region between the seesaw scales: $M_1 < \mu < M_3$, where M_3 is the heaviest RH neutrino.

3). Region above the seesaw scales: $\mu > M_3$.

In the regions 2) and 3) the key new feature is that some or all RH neutrinos are not decoupled and therefore the neutrino Yukawa couplings, Y_ν, contribute to the running in addition to Y_e. The couplings now Y_ν run. The term $C_\nu Y_\nu^\dagger Y_\nu$, where $C_\nu = 0.5$ in SM and $C_\nu = 1$ in MSSM, should be added to the RGE eq.(46). Also α should be modified.

The couplings Y_ν can be large - of the order 1 both in SM and MSSM independently of $\tan\beta$. Therefore in general the RGE effects due to Y_ν are large.

Another important feature is that the matrices Y_ν and Y_e can not be made both diagonal, *e.g.*, Y_ν is non-diagonal in the flavor basis where $Y_e = diag$. This means that RGE running due to Y_ν generates the flavor transitions and therefore leads to rotation of the flavor basis. This running can produce flavor mixing even if initial (at boundary) mixing matrix is proportional to unity [50].

In the region above the seesaw scales the running has similar features to those in the range (1). In particular, similar enhancement factors appear in the case of degenerate or partially degenerate spectra. Also CP-violating phases influence running substantially leading in certain cases to damping of the enhancement. The 1-2 angle can undergo the strongest renormalization.

Let us note some interesting possibilities. For the degenerate spectrum and certain values of phases, running above the seesaw scales, $(10^{14} - 10^{16})$ GeV can reduce θ_{12} from 45° down to $\sim 30^\circ$, thus explaining deviation of the lepton mixing from bi-maximal. It can correct the QLC-1 relation reducing θ_{12} at low energies. Running of other two angles is substantially weaker. Renormalization effects in two other region can be small, *e.g.*, due to small $\tan\beta$.

For the hierarchical mass spectrum ($m_1 < 0.01$ eV) the RGE induced changes of the mixing parameters are relatively small: $\Delta\theta_{12} < (1-2)^\circ$ (though it may be relevant for the QLC relation), $\Delta \sin^2\theta_{13} < 3 \cdot 10^{-5}$ and $\Delta \sin^2\theta_{23} < 0.02$. Mass squared difference Δm^2_{12} can decrease due to running between and above seesaw scales by factor of 2 for partially degenerate spectrum, *etc.* [50].

In the region between seesaw scales one or two RH neutrinos decouple. The effective neutrino mass matrix has two different contributions: $\mathcal{M}_\nu = \mathcal{M}_{run} + \mathcal{M}_{dec}$ - d=5 type term from the decoupled states, $\mathcal{M}_{dec}$, and the running term, $\mathcal{M}_{run} = \tilde{Y}_\nu^T(\mu) M_R^{-1}(\mu) \tilde{Y}_\nu(\mu)$, where where $\tilde{Y}_\nu$ is the submatrix (3×2 or 3×1) of the Yukawa couplings for "undecoupled" RH neutrinos. In non-supersymmetric models these two contributions renormalize differently due to vertex diagrams (in the SM case) which include RH neutrino propagator. This can change substantially the observables, *e.g.*, leading to $\Delta\theta_{12} \sim 10^\circ$ even for the hierarchical spectrum [50].

D. Other realizations of seesaw

If it turns out that the scale of $B-L$ symmetry is in the TeV range, as for example, in a class of string models discussed recently [106], small neutrino mass can be understood by a double seesaw mechanism [107] where in addition to the right handed neutrino, N, one postulates the existence of a singlet neutrino S. The symmetries of the model are assumed to be such that the Majorana mass of N as well as the coupling of S to the lepton doublet are forbidden. We then have a neutrino mass matrix in the basis (ν, N, S) of the form:

$$M = \begin{pmatrix} 0 & m_D^T & 0 \\ m_D & 0 & M \\ 0 & M^T & \mu \end{pmatrix}. \quad (79)$$

For the case $\mu \ll M \approx M_{B-L}$, where M_{B-L} is the $B-L$ breaking scale, this matrix has one light and two heavy quasi-degenerate states for each generation. The mass matrix of light neutrinos is given by

$$\mathcal{M}_\nu \sim m_D^T M^{T-1} \mu M^{-1} m_D. \quad (80)$$

There is a double suppression by the heavy mass compared to the usual seesaw mechanism and hence the name double seesaw. One important point here is that to keep $\mu \sim m_D$, one also needs some additional gauge symmetries, which often are a part of the string models.

Another possibility which is motivated by the fact that the required masses of the RH neutrinos are at somewhat smaller scale than the GUT scale is that the RH neutrinos themselves get the mass via seesaw mechanism generated by N and S. That would correspond to $\mu \gg M$ in the eq.(79), so that

$$M_R = -M \mu^{-1} M^T. \quad (81)$$

For $\mu \sim M_{Pl}$ and $M \sim M_{GUT}$ that gives the required masses of the RH neutrinos. In particular eq.(81) can produce strong hierarchy of masses. The formula for the light masses will be the same as in eq.(80).

It has been noted [108] that if there is parity symmetry in models that implement the double seesaw mechanism, then the 13 and 31 entries of the above neutrino mass matrix get filled by small seesaw suppressed masses. This leads to

$$M = \begin{pmatrix} 0 & m_D^T & M^T\epsilon' \\ m_D & 0 & M \\ M\epsilon' & M^T & \mu \end{pmatrix}, \tag{82}$$

where $\epsilon' \simeq v_{wk}/V_0$ and V_0 is of the order the mass M in eq.(82). The neutrino mass in this case is given by:

$$\mathcal{M}_\nu = m_D^T M^{T-1}\mu M^{-1} m_D - (m_D + m_D^T)\epsilon'. \tag{83}$$

The last contribution, linear in the Dirac masses, is called the seesaw type III. There have been few applications of this mechanism to model building [109].

There are also other variations on the seesaw theme for instance having two right handed neutrinos rather than three. Two RH neutrinos is the minimum number that will give a realistic spectrum for neutrinos after seesaw mechanism. There are schemes where new symmetries beyond the standard model can realize such a possibility [46, 110]. For instance, if we supplement the standard model by a local $SU(2)_H$ horizontal symmetry that acts on the first two generations, then global anomaly freedom only requires that there be two right handed neutrinos transforming as a doublet under $SU(2)_H$. This model leads to a 3×2 seesaw and has features similar to the two RH dominance models [111].

E. Seesaw and large lepton mixing

1. *See-Saw enhancement of mixing*

[112, 113]. Can the same mechanism which explains the smallness of the neutrino mass, that is, seesaw also explain the large lepton mixing, so that eventually large mixing originates from zero neutrino charges and Majorana nature?

The idea is that due to the (approximate) quark-lepton symmetry, or GUT, the Dirac mass matrices of the quarks and leptons have all the same (or similar) structure: $m_D \sim m_{up} \sim m_l \sim m_{down}$ leading to zero (small) mixings in the first approximation. Due to non-diagonal mass matrix of RH neutrinos, M_R, which has no analogue in the quark sector, the seesaw mechanism leads to non-zero lepton mixing already in the lowest order.

The problem with this scenario is the strong hierarchy of the quark and charged lepton masses. Indeed, taking the neutrino Dirac masses as $m_D = diag(m_u, m_c, m_t)$ in the spirit of GU, we find that for a generic M_R the see-saw type I formula produces strongly hierarchical mass matrix of light neutrinos with small mixings. The mixing becomes large only for special special structure of M_R which compensates the strong hierarchy in m_D.

Two different possibilities are [112]:

- strong (nearly quadratic) hierarchy of the RH neutrino masses: $M_{iR} \sim (m_{iup})^2$ which can be naturally reproduce by the double seesaw; and
- strongly off-diagonal structure of M_R (pseudo Dirac structures) like

$$M_R = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix} \tag{84}$$

  which implies certain symmetry. Alternatively, (12), (21) and (33) elements can be non-zero. Interesting consequence of these structures is that the pair of the RH neutrinos turns out to be nearly degenerate which can lead to resonant leptogenesis.

In the three neutrino context both possibilities can be realized simultaneously, so that the pseudo Dirac structure leads to maximal 2-3 mixing, whereas the strong hierarchy $A \ll B$ enhances the 1-2 mixing [114].

There are several alternatives to the seesaw enhancement.

2. Large mixing from type II see-saw.

In general, the structure of neutrino mass matrix generated by the type II (triplet) see-saw is not related to structures of matrices of other fermions and it can produce large mixing (see sect. 7). In some particular cases, however, the relations can appear leading to interesting consequences (see below).

3. Single RH neutrino dominance

[115]. The large neutrino mixing and relatively strong mass hierarchy implied by the solar and atmospheric neutrino data can be reconciled if only one RH neutrino gives the

dominant contribution to the see-saw. (This leads to the (2-3) submatrix of m_ν with nearly zero determinant.) There are two different realizations of this possibility. In one case the large mixing originates from the large mixing in the Dirac neutrino mass matrix m_D: two LH neutrinos have nearly equal couplings to the dominating RH component. Suppose that $(m_D)_{23} \approx (m_D)_{33} = m$, $(m_D)_{13} = \lambda m$ ($\lambda \approx 0.2$) and all other elements of m_D are much smaller. Then if only $(M^{-1})_{33}$ is large in the inverted matrix, the see-saw leads to the mass matrix with dominant $\mu - \tau$ block.

The mechanism can also be extended to enhance 1-2 mixing it requires the so called sequential dominance related to the second RH neutrino [116].

In another version, the dominance is realized when two RH neutrinos are much heavier than the third (dominating) one and no large mixing in m_D appears. This is equivalent to the strong mass hierarchy case of the see-saw enhancement mechanism. A realization requires $(m_D)_{22} \approx (m_D)_{23} \ll (m_D)_{33}$, and $(M^{-1})_{22}$ being the dominant element.

It may happen that the enhancement of the mixing is not related to the seesaw mechanism at all being, *e.g.*, of the radiative origin. Let us consider the following possibility.

4. Lopsided models

[117]. Large lepton mixing in these models follows from the charge lepton mass matrix in the symmetry basis which should be left-right non-symmetric. This does not contradict the Grand Unification since in GUT models such as SU(5) the LH components of leptons are unified with the RH components of quarks: $5 = (d^c, d^c, d^c, l, \nu)$. Therefore large mixing of the LH leptonic components is accompanied by large mixing of the RH d-quarks which is unobservable. Introducing the Dirac mass matrix of the charged leptons with the only large elements $(m_l)_{33} \sim (m_l)_{23}$ in the basis where neutrino mass matrix is nearly diagonal, one obtains the large 2-3 lepton mixing. This scenario can also be realized in $SO(10)$, if the symmetry is broken via $SU(5)$. A double lopsided matrix for both large mixings (solar and atmospheric) is also possible.

F. Screening of Dirac structure

The quark - lepton symmetry manifests itself as certain relation (similarity) of the Dirac mass matrices of quarks and leptons, and this is the origin of problems in explanation of strong difference of mixings and possible existence of the "neutrino symmetries". However, in the context of double seesaw mechanism the Dirac structure in the lepton sector can be completely eliminated - "screened"[112, 118, 119] thus opening new possibilities.

Indeed, the double seesaw mechanism leads to the light neutrinos mass matrix given in eq.(80). Suppose that due to certain family symmetry or Grand Unification (which includes also new singlets S) the two Dirac mass matrices are proportional to each other:

$$M_D = A^{-1} m_D, \quad A \equiv v_{EW}/V_{GU}. \tag{85}$$

In this case the Dirac matrices cancel in (80) and we obtain

$$m_\nu = A^2 M_S. \tag{86}$$

That is, the structure of light neutrino mass matrix is determined directly by M_S and does not depend on the Dirac mass matrix. Here the seesaw mechanism provides the scale of neutrino masses but not the flavor structure of the mass matrix. It can be shown that at least in SUSY version radiative corrections do not destroy screening [118].

Structure of the light neutrino mass matrix is given up to small corrections by M_S which can be related to some new physics at, *e.g.*, the Planck scale. In particular,

1). M_S can be the origin of "neutrino" symmetry;

2). $M_S \propto I$ leads to the quasi-degenerate spectrum of light neutrinos;

3). M_S can be the origin of bi-maximal or maximal mixing thus leading to the QLC relation [83] if the charged lepton mass matrix generates the CKM rotation (QLC-1).

In general, screening allows one to "automatically" reconcile the $q - l$ symmetry with strong difference of mixings of leptons and quarks.

G. Seesaw: tests and applications

A major problem in neutrino physics is to find ways to test the proposed mechanisms and scenarios of neutrino mass generation. The seesaw scenarios are related to physics at very high energy scales which can not be achieved by the direct studies. Furthermore, it is practically impossible to reconstruct the right handed neutrino mass matrix from the low energy observables [120] without additional assumptions like involvement of only two RH neutrinos [121], *etc.*.

The situation can change if the seesaw mechanism is embedded in into bigger picture so that one will be able to test the whole context. This will allow to connect neutrinos with other phenomena and observables. Moreover, some parameters of the seesaw mechanism can be determined (see sect. 7).

The hope is that in future on the basis of certain models we will be able to make predictions with *very small uncertainties* which can be tested in *precision* measurements. This will then provide a direct test of the model.

In what follow we will describe briefly some connections of the seesaw with other phenomena which can help to check the mechanism.

1. Origin of matter.

A very interesting aspect of the seesaw mechanism is the possibility that the heavy right handed neutrino decays and CP violation in the lepton sector may provide a way to understand the origin of matter - baryon asymmetry of the Universe [122].

The original scenario consists of out of equilibrium CP violating decays of the RH neutrinos $N \to l + H$ which lead to production of the lepton asymmetry. This asymmetry gets partly transformed to baryon asymmetry by sphaleron processes (which conserve B - L but violate B+L).

One of the goals of this discussion would be to learn about the right handed neutrinos and the nature of leptonic CP violation from the condition of successful leptogenesis [123]. While there are many possible ways to achieve successful leptogenesis, *e.g.*, resonant leptogenesis, non-thermal leptogenesis, *etc.*, we will restrict our discussion to the simplest case of hierarchical pattern for right handed neutrino masses, *i.e.* $M_1 < M_{2,3}$ and thermal leptogenesis and outline their consequences for neutrinos.

The first implication of leptogenesis for right handed neutrino spectrum comes from the out of equilibrium condition for their decay:

$$\Gamma_i \leq H(M_i) \simeq \sqrt{g_*}\frac{M_i^2}{M_{P\ell}}, \tag{87}$$

where Γ_1 is the decay rate

$$\Gamma_i \sim \frac{(Y_\nu Y_\nu^\dagger)_{ii} M_i}{8\pi} \tag{88}$$

and M_i is the mass of the ith RH neutrino $H(M_i)$ is the expansion rate of the Universe in the epoch with temperature $T \sim M_i$, g_* is the number of relativistic degrees of freedom in the epoch T. This condition leads to the lower bound on the mass of the RH neutrino

$$M_i \geq \frac{M_{P\ell}|Y_{\nu,ik}|^2}{8\pi\sqrt{g_*}}. \tag{89}$$

One can get some idea about required values of masses M_i, *e.g.* assuming the up quark Yukawa couplings as a guideline for the Dirac neutrino couplings: $Y_{\nu,ik} \sim m_{u_i}/v_{wk}$. Then the out of equilibrium conditions eq.(89) would imply that $M_1 \geq 10^7$ GeV, $M_2 \geq 10^{12}$ GeV and $M_3 \geq 10^{16}$ GeV. So, the lepton asymmetry is assumed to be produced by the decay of the lightest N_1.

In this scenario the baryon asymmetry, $\eta_B \equiv n_B/s$, where n_B is the number density of baryons and s is the entropy density, can be written as [123]

$$\eta_B = \frac{8}{23}\frac{n_1}{s}\epsilon_1\kappa_1. \tag{90}$$

Here n_1 is the number density of the RH neutrinos, ϵ_1 is the lepton asymmetry produced in the decay of N_1 and κ_1 is the wash out factor which describes the degree of out-of-equilibrium condition; the factor 8/23 is the fraction of the $L-$ $(B-L)$ asymmetry which is converted to the baryon asymmetry by sphalerons. The quantities n_1, ϵ_1 and κ_1 all the functions of the RH neutrino masses and the Dirac type Yukawa couplings. So the bounds on the neutrino parameters can be obtained from simultaneous analysis these factors [123, 124].

Some more information can be gained by analyzing the magnitude of the lepton asymmetry ϵ_1 in terms of the Yukawa couplings Y_ν:

$$\epsilon_1 = \frac{-3}{16\pi(Y_\nu Y_\nu^\dagger)_{11}}\sum_{k\neq 1} Im\left[(Y_\nu Y_\nu^\dagger)^2_{1k}\right] f\left(\frac{M_k^2}{M_1^2}\right), \tag{91}$$

where in the case of hierarchical mass spectrum for the RH neutrinos, $x \equiv M_k^2/M_1^2 \gg 1$, the function $f(x)$ can be approximated as $f(x) \simeq -3/(2\sqrt{x})$ simplifying the above expression.

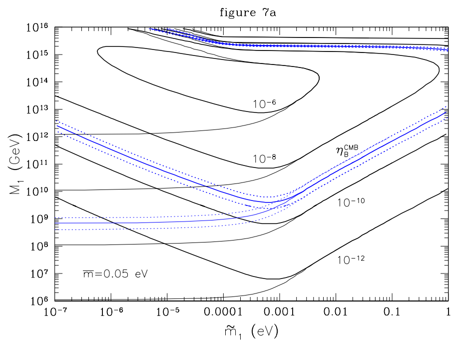

FIG. 9: Contour plot of the baryon to photon ratio produced in thermal leptogenesis, as a function of M_1 and $\tilde{m}_1$, from [124]. The decay asymmetry ϵ_1 was taken to be 10^{-6}. The three (blue) close-together lines are the observed asymmetry. The horizontal contours, for small $\tilde{m}_1$ assume a thermal N_1 abundance as initial condition.

Since $\eta_B = 6.3 \cdot 10^{-10}$, and κ_1 is roughly of order 10^{-3} (although it strongly depends on parameters in the model), we must have $\epsilon_1 \geq 10^{-6}$, which puts according to (91) a constraint on the flavor structure of Y_ν.

Another constraint on the Y_ν follows from consideration of the decay rate of N_1, which can be written as (88)

$$\Gamma_1 = \frac{\tilde{m}_1 M_1^2}{8\pi v_{wk}^2}, \qquad \tilde{m}_1 \equiv \frac{(Y_\nu Y_\nu^\dagger)_{11} v_{wk}^2}{M_1}. \tag{92}$$

This rate controls the initial abundance of N_1 and also the out-of-equilibrium condition. Successful leptogenesis restricts the values of $\tilde{m}_1$ and M_1 as shown in Fig. 9.

According to results of fig. 9 this standard scenario implies a lower bound on the lightest RH neutrino mass $M_1 \geq 10^8$ GeV and correspondingly gives the upper bound on the light neutrino masses thereby essentially excluding the degenerate spectrum for type I seesaw case.

These constraints can be avoided/weakened if one assumes type II seesaw [125] and/or some specific flavor structures of the Yukawa couplings [126]. The bound can be also weakened in the case of strong degeneracy of RH neutrino masses $M_1 \approx M_2$ which leads to enhancement of the asymmetry ϵ_1 (resonance leptogenesis [127]) and therefore allows for smaller k_1. Consequently, the washout (out of equilibrium) conditions relaxes the bound on

M_1.

It was proposed recently that cosmological density perturbations can be generated by the inhomogeneous decay of right-handed neutrinos [128]. That requires coupling of the RH neutrinos with a scalar field whose fluctuations are created during inflation.

2. Lepton flavor violation as tests of seesaw.

Once one includes the right handed neutrinos N in the standard model so that neutrinos acquire masses and mixings, the lepton flavor changing effects such as $\mu \to e+\gamma$, $\tau \to e, \mu+\gamma$, *etc.* appear. However, a simple estimate of the one loop contribution to such effects leads to an unobservable branching ratios (of order $\sim 10^{-40}$).

The situation, however changes drastically as soon as the seesaw mechanism is embedded into the supersymmetric models. Flavor changing effects arise from the mixings among sleptons (superpartners of leptons) of different flavors caused by the renormalization group corrections which via loop diagrams lead to lepton flavor violating (LFV) effects at low energies [129].

The way this happens is as follows. In the simplest N=1 supergravity models, the supersymmetry breaking terms at the Planck scale are taken to have only few parameters: a universal scalar mass m_0, universal A terms, and one gaugino mass $m_{1/2}$ for all three types of gauginos. Clearly, a universal scalar mass implies that at Planck scale, there is no flavor violation anywhere except in the Yukawa couplings. However, as we extrapolate this theory to the weak scale, the flavor mixings in the Yukawa interactions induce flavor violating scalar mass terms. In the absence of neutrino masses, the Yukawa matrices for leptons can be diagonalized so that there is no flavor violation in the lepton sector even after extrapolation down to the weak scale. On the other hand, when neutrino mixings are present, there is no basis where all leptonic flavor mixings can be made to disappear. In fact, in the most general case, of the three matrices: Y_ℓ - the charged lepton coupling matrix, Y_ν - RH neutrino Yukawa coupling and M_R - the matrix characterizing the heavy RH neutrino mixing, only one can be diagonalized by an appropriate choice of basis and the flavor mixing in the other two remains. In a somewhat restricted case where the right handed neutrinos do not have any interaction other than the Yukawa interaction and an interaction that generates the

Majorana mass for the right handed neutrino, one can only diagonalize two out of the three matrices (*i.e.*, Y_ν, Y_ℓ and M_R). Thus, there will always be lepton flavor violating terms in the basic Lagrangian, no matter what basis one chooses. These LFV terms can then induce mixings between the sleptons of different flavor and lead to LFV processes.

In the flavor basis, searches for LFV processes such as $\tau \to \mu + \gamma$ and/or $\mu \to e + \gamma$ can throw light on the RH neutrino mixings and/or family mixings in m_D, as has already been observed.

Since in the absence of CP violation, there are at least six mixing angles (nine if m_D is not symmetric) in the seesaw formula and only three are observable in neutrino oscillation, to get useful information on the fundamental high scale theory from LFV processes, it is often assumed that M_R is diagonal so that one has a direct correlation between the observed neutrino mixings and the fundamental high scale parameters of the theory. The important point is that the flavor mixings in Y_ν then reflect themselves in the slepton mixings that lead to the LFV processes via the RGEs.

To give a typical estimate of the magnitude of lepton flavor violation in seesaw models, we can make a simple ansatz of equal RH neutrino masses and assume CP conservation. The slepton mixing defined by $\Delta_{LL,ij} \equiv \frac{\delta m^2_{ij}}{m^2_0}$ can be estimated from the renormalization group equation to be

$$\Delta_{LL,ij} \;=\; \frac{3}{8\pi^2}[Y^\dagger_\nu Y_\nu]_{ij} ln\frac{M^2}{v^2_{wk}} \simeq \frac{1}{4\pi^2}\frac{M(m_\nu)_{ij}}{v^2_{wk}} ln\frac{M_{P\ell}}{M}, \tag{93}$$

where M is the seesaw scale. Using $M \sim 10^{13}$ GeV or so, one finds that the branching ratios for $\mu \to e+\gamma$ and $\tau \to \mu+\gamma$ depend on the slepton masses like m^{-4} and go down as slepton masses increase as can be seen from

$$R(\ell_j \to \ell_i + \gamma) \equiv \frac{B(\ell_j \to \ell_i + \gamma)}{B(\ell_j \to \ell_i + \nu_j + \bar{\nu}_i)} \simeq \frac{3\alpha_{em}(c_1 g_1^2 + g_2^2)^2}{32\pi m^4_{sl} G^2_F} \left(\Delta_{LL,ij}\right)^2 \tan^2\beta \; . \tag{94}$$

For the masses in the (200 - 500) GeV range, the $R(\mu \to e + \gamma)$ can be above 10^{-14}, a value which can be probed by the MEG experiment in progress [130]. Similarly for the same slepton masses, $R(\tau \to \mu + \gamma)$ can be in the range of $10^{-9} - 10^{-8}$ or so.

VII. NEUTRINO MASS AND NEW PHYSICS: "TOP-DOWN"

A. Neutrino mass and Grand Unification.

One of the major ideas for physics beyond the Standard Model is supersymmetric grand unification (SUSY GUT) [131]. It is stimulated by a number of observations that are in accord with the general expectations from SUSY GUT's: (i) A solution to the gauge hierarchy problem *i.e.* why $v_{\rm wk} \ll M_{\rm Pl}$; (ii) unification of electro-weak, *i.e.*, $SU(2)_L \times U(1)_Y$ and strong $SU(3)_c$ gauge couplings assuming supersymmetry breaking masses are in the TeV range, as would be required by the solution to the gauge hierarchy; (iii) a natural way to understand the origin of electroweak symmetry breaking.

As noted earlier, closeness of the gauge coupling unification scale of about 10^{16} GeV and an estimate of the seesaw scale from atmospheric neutrino data of $M_3 \sim 10^{15}$ GeV suggests that seesaw scale could be the GUT scale itself. So the smallness of neutrino mass goes quite well with the idea of supersymmetric grand unification. However, in contrast with the items (i) through (iii) listed above, the abundance of information for neutrinos makes it a highly nontrivial exercise to see whether the neutrino mixings indeed fit well into simple SUSY GUTs.

The simplest GUT group is SU(5). Since the basic matter representations of SU(5), $\bar{\mathbf{5}}\oplus\mathbf{10}$, do not contain the right handed neutrino, one must extend the model by adding three right-handed neutrinos, one per generation. The problem then is that Majorana mass of the gauge singlet right handed neutrino is unconstrained and can be same as the Planck mass which will make it difficult to accommodate the neutrino data. The right handed neutrino mass fine tuning question, *i.e.*, why $M_R \ll M_{Pl}$ arises again. However, if one includes the **15** dimensional Higgs boson, then Yukawa coupling $\bar{\mathbf{5}}_m\bar{\mathbf{5}}_m\mathbf{15}_H$ in the superpotential leads to the coupling $LL\Delta_L$ where Δ_L is the $SU(2)_L$ triplet in the **15**-Higgs. The Δ^0_L field acquires a vev of order $v^2_{wk}/\lambda M_U$ [132], where λ is a typical coupling parameter for the Higgs fields among themselves and M_U is the scale of grand unification. This leads to neutrino masses (as in type II seesaw) of the right order to explain the data.

On the other hand, if one considers the SO(10) group [75], then its basic spinor representation contains the right-handed neutrino automatically along with the other fifteen fermions of the Standard Model (for each family). In order to give a mass to the right handed neutrino, one must therefore break SO(10) symmetry (more precisely, the B-L subgroup of SO(10)). This naturally solves the right handed neutrino mass fine tuning problem. Thus, one could argue that small neutrino masses have already chosen $SO(10)$ GUT as the most natural way to proceed beyond the Standard Model. Therefore $SO(10)$ has rightly been the focus of many attempts to understand neutrino mixings.

The $SO(10)$ SUSY GUT models can be broadly classified into two classes. One class of models that employ **16**-Higgs representation to give mass to the right handed neutrinos and another that employs **126** Higgs. We outline below their major features and differences.

As noted, one of the features that distinguishes $SO(10)$ from SU(5) is the presence of local $B-L$ symmetry as a subgroup, and the two classes of the $SO(10)$ models mentioned above differ in the way the $B-L$ symmetry is broken: breaking by $\mathbf{16}_H$ Higgs field gives $\Delta(B-L) = 1$ whereas **126** leads to $\Delta(B-L) = 2$. In the first case the right-handed neutrino mass necessarily arises out a nonrenormalizable coupling whereas in the second case it arises from a renormalizable one. Secondly, the breaking of $B-L$ by **16** Higgs necessarily leads to low energy MSSM with R-parity breaking so that the model cannot have cold dark matter without additional assumptions such as matter parity which forbids specific couplings such as $(\mathbf{16}_m)^3\mathbf{16}_H$, where $\mathbf{16}_m$ stands for the matter spinor.

On the other hand, **126** breaking of $B-L$ preserves R-parity at low energies, so that the low energy MSSM that derives from such an $SO(10)$ has a natural dark matter candidate, *i.e.* the lightest SUSY particle.

Since $SO(10)$ contains the left-right symmetric group as a subgroup, it can either have a type II or type I seesaw formula for neutrino masses depending on the details of symmetry breaking and parameter ranges of the theory. For instance, in the $\mathbf{16}_H$ based models, the type II seesaw term is negligible and therefore the neutrino masses are dictated by type I seesaw formula. In contrast, in **126** Higgs models, the neutrino mass can be given either by the first term or the second term in the general seesaw formula, or both.

1. A minimal **126***-based* **SO(10)** *model.*

Since $\mathbf{16} \otimes \mathbf{16} = \mathbf{10} \oplus \mathbf{120} \oplus \mathbf{126}$, the natural minimal model is to consider all the three Higgs fields and have them couple to the matter **16**. A simpler model contains only $\mathbf{10} \oplus \mathbf{126}$ in which case there are only two Yukawa coupling matrices: (i) h for the **10** Higgs and (ii) f for the **126** Higgs [133]. SO(10) has the property that the Yukawa couplings involving the **10** and **126** Higgs representations are symmetric. Therefore in order to reduce the number of free parameters, one may assume that Yukawa couplings are CP conserving and CP violation arises from other sectors of the theory (*e.g.* squark masses). In a basis one of these two sets of Yukawa coupling matrices is diagonal, and the Yukawa sector will have only nine parameters. Noting the fact that the (2,2,15) submultiplet of $\mathbf{126}_H$ as well as (2,2,1) of $\mathbf{10}_H$ each have a pair of standard model doublets that contributes to charged fermion masses, one can write the quark and lepton mass matrices as follows [133]:

$$M_u = h\kappa_u + fv_u, \qquad M_d = h\kappa_d + fv_d, \tag{95}$$

$$M_\ell = h\kappa_d - 3fv_d, \qquad m_D = h\kappa_u - 3fv_u, \tag{96}$$

where $\kappa_{u,d}$ are the vev's of the up and down standard model type Higgs fields in the $\mathbf{10}_H$ multiplet and $v_{u,d}$ are the corresponding vev's for the same doublets in $\mathbf{126}_H$. Note that there are 13 parameters in the equations above (nine parameters in the Yukawa couplings noted above and four vacuum expectation values for the four MSSM doublets in the **10** and **126** Higgs fields) and there are 13 inputs (six quark masses, three lepton masses and three quark mixing angles and weak scale). Thus, all parameters of the model that go into fermion masses are determined.

To generate the light neutrino masses, we use the seesaw formula in eq.(78), where the f is nothing but the same $\mathbf{126}_H$ Yukawa coupling as above. Thus all parameters that give neutrino mixings except an overall scale are determined[134].

To see how large mixings arise in this model let us assume that the seesaw type II gives dominant contribution to the neutrino mass, so that

$$m_\nu \propto f. \tag{97}$$

Then using eq.(97), we obtain

$$m_\nu = c(M_d - M_\ell). \tag{98}$$

By a choice of fermion basis, we can have M_d to be diagonal, so that all the quark mixing effects are then in the up quark mass matrix, *i.e.* $M_u = U^T_{CKM} M^d_u U_{CKM}$. Note further that the minimality of the Higgs content leads to the following sum rule among the mass matrices:

$$kM_\ell \;=\; rM_d + M_u. \tag{99}$$

We then find using the known mass and mixing pattern for quarks that

$$M_{d,\ell} \;\approx\; m_{b,\tau}\begin{pmatrix}\lambda^3 & \lambda^3 & \lambda^3\\ \lambda^3 & \lambda^2 & \lambda^2\\ \lambda^3 & \lambda^2 & 1\end{pmatrix}, \tag{100}$$

where $\lambda \sim \sin\theta_C = 0.22$ and the matrix elements are supposed to give only the approximate order of magnitude. An important consequence of the relation between the charged lepton and the quark mass matrices in eq.(99) is that the charged lepton contribution to the neutrino mixing matrix, *i.e.* $U_\ell \simeq \mathbf{1} + O(\lambda)$ or close to identity matrix. As a result large neutrino mixings must arise predominantly from neutrino mass matrix given by the type II seesaw formula. In the actual calculations of course the charged lepton mixings are also taken into account. The phenomenological fact that $m_b - m_\tau \approx m_\tau\lambda^2$ for a wide range of values of $\tan\beta$ now implies that, the neutrino mass matrix takes roughly the form

$$\mathcal{M}_\nu \;= c(M_d - M_\ell) \approx\; m_0\begin{pmatrix}\lambda^3 & \lambda^3 & \lambda^3\\ \lambda^3 & \lambda^2 & \lambda^2\\ \lambda^3 & \lambda^2 & \lambda^2\end{pmatrix}, \tag{101}$$

where except for numbers of order one, the entire neutrino mass matrix is characterized by the Cabibbo angle alone. It is easy to see that both θ_{12} (the solar angle) and θ_{23} (the atmospheric angle) are now large [135, 136].

The main point illustrated by this model is that the large neutrino mixings need not be a consequence of symmetries but rather could arise dynamically out of $b-\tau$ unification at high scale. Note that this requires the choice of small f_{33} which was however made to fit the quark sector and not to "fix" the neutrino mixings. Of course, one must understand the flavor structure of the h and f Yukawa couplings (*e.g.* why f_{33} is so small) from a higher scale theory.

There are various ways to incorporate CP violation in these models (see [137, 138]). One could simply assume that the Yukawa couplings are complex. However in this case the simple connection between $b-\tau$ unification and large neutrino mixing is lost. On the other hand, one could assume that all Higgs representations that can couple to matter spinors *i.e.*

$\mathbf{10} \oplus \mathbf{120} \oplus \mathbf{126}$ are present [138]. If one imposes CP conservation in the full theory, then the $\mathbf{10}$ and $\mathbf{126}$ couplings become real whereas the $\mathbf{120}$ coupling is imaginary. In this case, the large atmospheric mixing arises via $b-\tau$ unification as before. In this model one can still predict θ_{13} and the Dirac phase δ. Furthermore this model also solves the SUSY CP problem.

An important consequence of this class of models is that the mixing parameter $\sin\theta_{13}$ is "large" being somewhere between $0.08-0.18$, which can be testable in the planned reactor and long baseline neutrino experiments.

The large Higgs representations as well as dominance of type II seesaw require that SO(10) symmetry is broken first to SU(5) and subsequently to MSSM [139], so that the gauge couplings remain perturbative close to the Planck scale.

2. $\mathbf{16}$*-Higgs based SO(10) models.*

The other class of SO(10) models for neutrinos that has been widely discussed in the literature includes $\mathbf{10}_H$, $\mathbf{16}_H$, $\overline{\mathbf{16}}_H$ and $\mathbf{45}_H$ only [140, 141, 142, 143]. An advantage of these models is that they use low dimensional Higgs multiplets. However, since in these models the only renormalizable term is the $\mathbf{16}_m\mathbf{16}_m\mathbf{10}_H$, this can neither explain the observed quark and lepton masses nor can it explain the neutrino masses. One has to therefore include higher dimensional operators in the Yukawa coupling such as $\mathbf{16}_m\mathbf{16}_m\mathbf{16}_H\mathbf{16}_H$, $\mathbf{16}_m\mathbf{16}_m\overline{\mathbf{16}}_H\overline{\mathbf{16}}_H$, $\mathbf{16}_m\mathbf{16}_H\mathbf{16}_m\mathbf{16}_H$, $\mathbf{16}_m\overline{\mathbf{16}}_H\mathbf{16}_m\overline{\mathbf{16}}_H$, $\mathbf{16}_m\mathbf{16}_m\mathbf{10}_H\mathbf{45}_H$, where $\mathbf{16}_m$ stands for various fermion generations. Of these, the first two give symmetric Yukawa couplings and the next two have no symmetry property and the last one can be both symmetric as well as antisymmetric. Since each coupling is a 3×3 matrix, there are many more free parameters in such models than observables. A strategy employed is to impose additional discrete symmetries to reduce the number of parameters. This and the fact that one can have large R-parity violation is a drawback for these models.

On the other hand these models have certain advantages: (i) it is possible to implement the doublet-triplet splitting in a simple way such that the low energy theory below the GUT scale is the MSSM and (ii) the threshold corrections to the gauge couplings are not excessive, so that no particular constraint on symmetry breaking is necessary for the gauge couplings to remain perturbative. Another distinction from the $\mathbf{126}$ based models is that the type II seesaw contribution to neutrino masses is small in this model. The MSSM Higgs doublets, *i.e.* $H_{u,d}$ fields are linear combinations of the doublets in $\mathbf{10}_H$, $\mathbf{16}_H$ and $\overline{\mathbf{16}}_H$. The right handed neutrino mass arises from the $\mathbf{16}_m \mathbf{16}_m \overline{\mathbf{16}}_H \overline{\mathbf{16}}_H$ couplings when $\tilde{\nu}^c$ component of $\overline{\mathbf{16}}_H$ acquires a vev. Typically, one uses the lopsided mechanism [117] to generate large atmospheric neutrino mixing.

Most **16**-Higgs based SO(10) models lead to small value for θ_{13}, although it is possible to have variations of the model which have bigger value for it [143].

B. Grand unification and flavor symmetry

While the hypothesis of grand unification goes naturally with the seesaw scale, the detailed flavor pattern *i.e.* hierarchical mass and mixing among quarks and large mixings for leptons is perhaps suggestive of some kind of flavor symmetries connecting different generations. A possible symmetry group for such models that unify quark and lepton flavor textures while at the same time implementing the seesaw mechanism could be, *e.g.*, SO(10)$\otimes G_{family}$, where G_{family} can be either SO(3), SU(3) [144] or SU(2)[146] or $U(1)$[145] group or a discrete group such as S_4[147], Z_2 or A_4, all of which have been attempted. The groups such as $SU(3), SO(3)$ as well as S_4 and A_4 have an advantage over the $U(1)$ and Z_2 groups since they have **3** dimensional representations into which the three families can fit unlike the other groups.

The main feature of these models is that in the case of abelian discrete group one can reproduce the flavor structure selecting the Yukawa couplings, whereas in the case of non-abelian ones, the problem shifts to VEV alignment and particular form of the scalar potential. This generally requires large number of Higgs fields with specific couplings. However, this appears to be a straightforward and promising direction for both quark-lepton and flavor unification and better models must be pursued.

It is also worth noting that if simpler models such as the minimal SO(10) model with **126** discussed above are experimentally favored, we must find a natural way based on some higher symmetry to generate the necessary form of the **126** Yukawa coupling f as in eq.(101).

Research along this line are mostly at an exploratory stage but it is probably fair to conjecture that such a unified approach is more likely to succeed if the neutrino mass hierarchy is established to be normal rather than inverted since the unification group connects all fermion textures. Furthermore, a strong signal of an underlying symmetry would be a degenerate spectrum. Examples of such symmetries which in conjunction with type II seesaw lead to degenerate spectrum have been discussed in the literature [147, 148].

C. Grand unification and proton decay

Since grand unified theories connect quarks and leptons, most such theories predict an unstable proton and therefore one could use proton decay as a signal of the specific nature of the grand unified theory. In supersymmetric theories since the dominant contribution to proton decay arises from dimension five operators which involve Yukawa couplings responsible for flavor structure of fermions, one may also hope to learn about the fermion textures from the proton decay modes.

In the context of SO(10) models, the predictions for proton decay have been studied in both **16** based [149] as well as **126** based models [150]. Both cases have typical predictions for distinguishing models, *e.g.* the cannonical $p \to \bar{\nu} K^+$ in the case of Ref.[149] and $n \to \pi^0 \bar{\nu}$ in the **126** case at an observable level.

It must however be stressed that a true test of grand unification would be the discovery of the gauge mediated proton decay mode $p \to e^+ \pi^0$ which is completely model independent. For supersymmetric theories however they are expected to be at the level of 10^{36} years or longer and are therefore beyond the reach of experiments with conventional technology. The situation is more hopeful for nonsupersymmetric theories.

D. Neutrinos and extra dimensions

One of the important predictions of string theories is the existence of more than three space dimensions. For a long time, it was believed that these extra dimensions are small and are therefore practically inconsequential as far as low energy physics is concerned. However, recent progress in the understanding of the nonperturbative aspects of string theories have opened up the possibility that some of these extra dimensions could be large without contradicting observations. In particular, models where some of the extra dimensions have sizes as large as a sub-millimeter and where the string scale M_* is in the few TeV range have attracted a great deal of phenomenological attention [151]. The basic assumption of these models, inspired by the D-branes in string theories, is that the space-time has a brane-bulk structure, where the brane is the familiar (3+1) dimensional space-time, with the standard model particles and forces residing in it, and the bulk consists of all space dimensions where gravity and other possible gauge singlet particles live. One could of course envision (3+d+1) dimensional D-branes where d-space dimensions have miniscule ($\leq$ TeV^{-1}) size. The main interest in these models has been due to the fact that the low string scale provides an opportunity to test them using existing collider facilities.

In general the extra dimensional theories can be divided into three broad classes: (i) very small size flat extra dimensions ($r \sim M_U^{-1}$ or so); (ii) large flat extra dimensions (i.e. $r \sim$ millimeter and (iii) warped extra dimensions of Randall-Sundrum type. Let us first discuss the second class of models.

In models with M_U^{-1} sized extra dimensions, one can implement the seesaw mechanism to generate neutrino masses. These models fit in very well with the conventional grand unified theories. These models have become popular as a way to providing an alternative resolution of the doublet-triplet splitting problem of grand unified theories via orbifold compactification[152]. As far as the flavor problem goes, if all the flavors are in the same brane, the presence of extra dimension does not distinguish between them and therefore does not throw any light on this issue. There are however models where different fermion generations are put in different locations in extra dimensions[153] which then leads to non-trivial flavor structure and a possible way to approach the flavor problem. Usually however extra assumptions such as symmetries are needed to get realistic models.

Coming to models with large extra D models (case (ii), a major challenge to them comes from the neutrino sector. There are several problems: (i) how to understand the small neutrino masses in a natural manner since the seesaw mechanism does not work here due to lack of a high scale; (ii) second problem is that if one considers only the standard model group in the brane, operators such as $LHLH/M_*$ could be induced by string theory in the low energy effective Lagrangian. For TeV scale strings this would obviously lead to unacceptable neutrino masses.

One mechanism suggested in Ref. [154] is to postulate the existence of gauge singlet neutrinos, ν_B, in the bulk which couple to the lepton doublets in the brane and additionally demand the theory to be invariant under the $B-L$ symmetry so that the higher dimensional operator $LHLH/M_*$ is absent. In four dimensions the Yukawa couplings and consequently

Dirac masses turn out to be suppressed by the ratio $M_*/M_{P\ell}$, where $M_{P\ell}$ is the Planck mass. The latter is now an effective parameter related to the volume of extra dimensions, $V_d = (2\pi)^d R_1 ... R_d$, and the fundamental scale as

$$M_{Pl}^2 = M_*^{2+d} V_d. \tag{102}$$

This suppression is sufficient to explain small neutrino masses and owes its origin to the large bulk volume in comparison with width of the brane $(1/M_*)^d$. The volume suppresses the effective Yukawa couplings of the Kaluza-Klein (KK) modes of the bulk neutrino to the brane fields.

Let us show appearance of the suppression using one extra dimension with coordinate y and radius R. The full action involving the $\nu_B(x, y)$ can be written as

$$\mathcal{S} = \int d^4x dy [i\bar{\nu}_B \gamma_\mu \partial^\mu \nu_B + i\bar{\nu}_{BL}(x, y)\partial_y \nu_{BR}(x, y) + \frac{h}{\sqrt{M_*}} \delta(y) \bar{L} H \nu_{BR}(x, y) h.c. \Big], \tag{103}$$

where $\mu = 0, 1, 2, 3$ and H denotes the standard model Higgs doublet. Expanding the bulk field in the Fourier series we obtain

$$\nu_R(x, y) = \sum_k \frac{1}{\sqrt{2\pi R}} \nu_R^{(k)} \cos \frac{ky}{R}, \tag{104}$$

where $\nu_R^{(k)}$ is the kth KK-mode and the prefactor follows from normalization of the wave function. Then according to eq.(104) the effective 4-dimensional Dirac coupling of the neutrino $\nu_R^{(k)}$ equals $\kappa = h/\sqrt{2\pi R M_*}$. Generalization for the case of d extra dimensions is straightforward: $2\pi R M_* \to V_d M_*^d$. Now using the relation between the four and $4 + d$ - dimensional Planck masses eq.(102) we get $\kappa = h\frac{M_*}{M_{P\ell}}$ independently of the number and configuration of extra dimensions. After standard model gauge symmetry breaking, this leads to a Dirac mass for the neutrino [154] given by

$$m = \frac{h v_{wk} M_*}{M_{P\ell}}. \tag{105}$$

For $M_* \sim (10 - 100)$ TeV, eq.(105) leads to $m \simeq (10^{-3} - 10^{-2})h$ eV. Since h is a five dimensional coupling, its value could perhaps be chosen ~ 10 in which case, we get neutrino mass in the range to be of interest in the discussion of neutrino oscillations. Furthermore, usual LH neutrino is mixed with all the KK modes of the bulk neutrino, with the same mixing mass $\sim \sqrt{2}m$. Since the kth KK mode has a mass $m_k = kR^{-1}$, the mixing angle is given by $\sqrt{2}mR/k$. Note that for $R \sim 0.1$ mm, this mixing angle is of the right order to be important *e.g.* in MSW transitions of solar neutrinos.

The above discussion can be extended in a very straight forward manner to the case of three generations. The simplest thing to do is to add three bulk neutrinos ascribing the generation label to all fermion fields. Now κ becomes a 3×3 matrix. One can first diagonalize this by rotating both the bulk and the active neutrinos. The mixing matrix then becomes the neutrino mixing matrix $\mathbf{U}$ discussed in the text. After this diagonalization one can perform the KK-expansion which leads to mixing of the active neutrinos and the bulk towers. There are now three mixing parameters, one for each mass eigenstate denoted by $\xi_i \equiv \sqrt{2}m_i R$ and mixing angle for each mass eigenstate to the kth KK-mode of the corresponding bulk neutrino is given by ξ/k.

In 4 dimensions the KK-modes of RH neutrinos will show up as sterile neutrinos. The main feature is that there is an infinite number of such neutrinos with increasing mass and decreasing mixing. This can lead to peculiar effects in neutrino oscillations. Till now, however, no effects are found which leads to the upper limits on ξ_i and hence the radius of the extra dimension, R, given a value of the neutrino mass m_i (or the coupling h). According to detailed analysis performed in [155] one has $R^{-1} \geq 0.02$ eV for hierarchical, ≥ 0.22 eV for inverted and ≥ 4.1 eV for degenerate neutrino spectrum. Generically, for all three cases the most stringent bound comes from the solar neutrino data (for the case of one extra dimension).

Coming to the third type of extra D models i.e. the Randall-Sundrum scenario, where one invokes a warped extra space dimension, understanding small neutrino masses is less straightforward and has not yet reached a level where its detailed phenomenological implications can be discussed although some interesting attempts have been made [156].

Essentially theories of extra dimensions provide us with qualitatively new mechanism of generation of the small *Dirac* neutrino mass. There are different scenarios, however their common feature can be called the overlap suppression: the overlap of wave functions of the left, $\nu_L(y)$, and right , $\nu_R(y)$ handed components in extra dimensions (coordinate y). The suppression occurs due to different localizations of the $\nu_L(y)$ and $\nu_R(y)$ in the extra space. The effective Yukawa coupling is proportional to the overlap. Thus, in the large flat extra dimensional scenario described above ν_L is localized in the brane which has volume $1/(M_*)^d$

in extra space, whereas ν_R propagates in the whole extra space volume V_n. So, the overlap equals the ratio of the two: $(1/M_*^d)/(V_d)$ which is precisely the factor we have discussed above. In the Randall-Sundrum scenario, ν_L and ν_R are localized into two different branes and the overlap of their wave functions is exponentially suppressed. Besides this, extra dimensions can be the origin of the light sterile neutrinos.

VIII. BEYOND THREE NEUTRINOS: STERILE NEUTRINOS AND NEW PHYSICS

An important part of our understanding of physics beyond the standard model involves a knowledge of whether there are only three light neutrinos $\nu_{e,\mu,\tau}$ or there are others. Known low energy particle physics as well as cosmology constrain the number and properties of any extra neutrinos. The fact that the measurement of the invisible Z-width at LEP is accounted for by the three known neutrinos to a very high degree of accuracy [157] implies that any extra light neutrino must not couple to the Z-boson and hence not the W boson either. Extra neutrinos are therefore called sterile neutrinos (ν_s).

Sterile neutrinos can communicate with usual active particles via Yukawa interactions. Non-zero VEV's of the corresponding scalar bosons generate the Dirac type mass terms which lead to mixing of active and sterile neutrinos. In turn this mixing may have important theoretical and phenomenological consequences.

A. Phenomenology of sterile neutrinos

Sterile neutrinos have very rich phenomenology. Possible existence of sterile neutrinos and their mixing have interesting consequences in particle physics, astrophysics and Cosmology.

Most of the studies however give the bounds on masses and mixing of these neutrinos (see [158] for recent review). In particular, if mixing of sterile neutrinos with active neutrinos is strong enough, they can come into equilibrium in the early universe and affect the big bang nucleosynthesis. Present data on primordial ^{4}He, ^{2}D and ^{7}Li abundances imposes constraint $N_{eff} < 1.5$ [159] on the effective number of sterile neutrinos which where in equilibrium in the epoch of nucleosynthesis. This in turn, leads to the bound on the active-sterile mixing θ_S as function of the mass m_S.

Strong bounds on parameters of sterile neutrinos come also from the structure formation in the Universe, from solar and supernova neutrinos, from studies of the electromagnetic radiation in the Universe (since sterile neutrinos have the radiative decay mode), *etc.*.

One may ask whether there is any need to introduce light sterile neutrinos. There are several reasons which are very suggestive:

(i) Interpretation of the excess of e^+n-events observed in LSND experiment [160] in terms of $\bar{\nu}_\mu - \bar{\nu}_e$ oscillations imply existence of one or several extra sterile neutrinos with masses $(1-5)$ eV [161, 162]. It should be stressed that such interpretation has its own problems. Furthermore it contradicts result of analysis of the large scale structure (LSS) in the Universe [163]. Another possibility is decay of the relatively heavy sterile neutrino with mass $\sim 0.01 - 0.1$ MeV [164]. Mini-BooNE experiment [165] is testing the LSND result.

(ii) Spherically asymmetric emission of sterile neutrinos with mass in the keV range during supernova collapses may explain the phenomenon of pulsar kicks [166].

(iii) Sterile neutrinos with mass $m_S \sim (1-3)$ kev were proposed to be the warm component of the Dark matter in the universe [167, 168]. However recent analysis of the LSS data is not compatible with this proposal [169].

(iv) Oscillations of these neutrinos in the Early Universe can be origin of the lepton asymmetry in the Universe [168, 170].

(v) Weak (statistically insignificant) indications of the presence of sterile states come from some solar neutrino data: low Homestake rate and absence of the upturn of the energy spectrum at low energies [171].

If for one reason or another the existence of sterile neutrinos are confirmed, it will be major revolution in the landscape of neutrino physics. We discuss some physics implications of this in this section, mainly focusing on the question of how to understand its lightness in the context of extensions of the standard model.

B. Sterile neutrinos and properties of active neutrinos

It may happen that sterile neutrinos have very small mixings for a given mass and therefore their astrophysical and cosmological effects are unobservable. In spite of this they can strongly influence the mass matrix of active neutrinos and therefore affect implications of the neutrino results for fundamental theory [172].

Suppose the active neutrinos acquire (*e.g.*, via seesaw) the Majorana mass matrix m_a. Consider one sterile neutrino, S, with Majorana mass m_S and mixing masses with active neutrinos $m_{aS}^T = (m_{eS}, m_{\mu S}, m_{\tau S})$. If $m_S \gg m_{aS}$, then after decoupling of S the mass matrix of active neutrinos becomes

$$m_\nu = m_a - \frac{m_{aS} m_{aS}^T}{m_S}, \tag{106}$$

where the last term is the matrix induced by S. Let us consider some possible effects.

The active-sterile mixing (induced matrix) can be the origin of large lepton mixing. Indeed, m_a may have usual hierarchical structure with small mixing. The mixing parameters m_{aS} can be chosen in such a way that the resulting matrix leads to large or maximal mixing [173].

The induced matrix can be the origin of particular neutrino symmetries. Consider a possibility that the coupling of a singlet field S with active neutrinos is universal: $m_{aS}^T = m_0(1,1,1) = m_2/\sqrt{3}$. Then the induced matrix has form:

$$\delta m_S = \frac{m_2}{3} D, \tag{107}$$

where D is the democratic matrix with all elements to be equal 1. Suppose that the original active neutrino mass matrix has structure

$$m_a = \frac{m_3}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{108}$$

Then the sum $m_\nu = m_a + \delta m_S$ reproduces the mass matrix for the tri/bimaximal mixing [172]. Second sterile neutrino can generate matrix eq.(108). Clearly this changes implications of the neutrino results which would require existence of sterile neutrinos, flavor blindness of their couplings, *etc.*. Since S is outside of the SM structure (with RH neutrinos) it may be easier to realize some particular symmetries for the induced matrix.

C. On the origin of sterile neutrinos.

Understanding origin of light sterile neutrinos is a big challenge. Their masses are not protected by the EW symmetry and some new physics (symmetries, dynamics) should exist to explain why they are light.

1. Mirror model for the sterile neutrino.

An interesting scenario for physics beyond the standard model has been discussed in literature where there is an identical copy of both the forces and matter present side by side with the known forces and matter. This new copy is called the mirror sector of the familiar universe. The mirror sector communicates with the familiar one only via gravitational interactions. This idea was originally proposed by Lee and Yang for the purpose of maintaining an exact parity symmetry in the full universe containing the mirror sector even though in each sector parity is violated in its weak interactions [174]. Such scenarios have recently emerged in the context string theories, where one has $E_8 \times E_8$ symmetry of matter and forces, with each E_8 acting on one 10- dimensional brane world and under mirror parity one brane goes into another. They are completely consistent with what is known about the low energy particles and forces as well as the standard big bang model of the universe if one assumes that in the process of evolution of the universe, the reheat temperature of the mirror sector is somewhat lower than the visible sector. Many interesting phenomenological consequences can follow in generic versions of such a theory at low energies such as neutrino oscillations, the dark matter of the universe *etc.*.

It was applied to the description of neutrino oscillation physics in [175], where it was noted that if sterile neutrinos indicated by the LSND results are confirmed, one of the ways to explain their lightness is to postulate the existence of the mirror sector of the universe in which case the mirror neutrinos can play the role of the sterile neutrinos and their lightness will follow from arguments similar to the familiar neutrinos, *e.g.*, via mirror seesaw. Electroweak symmetry can generate their mixings via operators of the form $LHL'H'/M$, where M could be the Planck mass representing the possibility that the two sectors mix via gravitational interactions. In general of course M could represent the mass of any standard model singlet particle. This model has also the potential to lead to sterile neutrinos in the keV range, that mix with known neutrinos.

2. Other possibilities.

Other models for the sterile neutrino include the possibility that it may be one of the standard model singlet fields present in string models [176], one of the extra singlet fermions in the E_6 models [177] or one of the seesaw right handed neutrinos which becomes massless due to leptonic symmetries [178] such as $L_e - L_\mu - L_\tau$ or $\mu - \tau$ exchange symmetry. A general feature of these models is that in the symmetry limit one of the SM singlet fermions remain massless which we can identify with the sterile neutrino. Its small mass and mixing with the active neutrinos is generated via the terms that break this symmetry.

IX. CONCLUSION

Recent discoveries in neutrino physics have opened up a new vista of physics beyond the standard model. In this review we have attempted to provide a glimpse of what we have learned from it and what the future experiments hold in terms how far this understanding can go. A broad theme is the appearance of new lepton flavor physics that was absent in the standard model with massless neutrinos with possibly important ramifications for the flavor physics of quarks. The main areas we have focussed on are: (i) understanding small neutrino masses; (ii) understanding the flavor structure of leptons that leads to large mixings and possible new symmetries implied by it and (iii) some possible implications of the existence new types of neutrinos. Of the several scenarios for understanding the small neutrino masses, the seesaw mechanism seems to have an advantage over others in many respects: (i) it provides a bridge to quark physics via grand unified theories; (ii) it gives a simple mechanism for understanding the origin of matter in the universe and (iii) has interesting low energy tests in the arena of lepton flavor violation, electric dipole moment of leptons, *etc.*. As far as the lepton flavor puzzle is concerned, while there are many interesting proposals, the final answer is far from clear and the next generation of experiments are very likely going to shed light on this issue. This process quite possibly will reveal new symmetries for leptons which in the broad framework of quark-lepton unification may throw new light on the quark flavor structure. We have summarized different ways to understand lepton mixings with and without the use of symmetries and discussed possible tests. If there appear evidence for new neutrino species mixing with known neutrinos, that will be a new surprise on top of the large mixing surprise and will be another revolution. It could raise questions such as: are there new quark species corresponding to the new neutrinos as well as what role they play in the evolution of the universe, *e.g.*, is there a mirror sector to the universe or are there extra dimensions?

Hopefully, we have made it clear that the field of neutrino physics is at an important cross-road in its evolution at the moment and further advance will depend on how we will answer the questions raised here. Some of the answers will very likely come from the proposed experiments that will test issues such as: is the neutrino its own antiparticle, how are the neutrino masses ordered, what is the absolute scale of neutrino mass. Further precision measurements of neutrino parameters, as well as searches for new (sterile) neutrinos are of fundamental importance.

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